Analyse asymptotique de jeux répétés à information incomplète.

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Introduction

Cadre et principaux résultats.

Nous présentons dans cette introduction quelques-uns des nombreux résultats existants portant sur le comportement asymptotique des jeux répétés à information incomplète d'un côté introduits par Aumann et Maschler [**3**]. Dans un premier temps, nous nous concentrons sur le comportement asymptotique de la valeur et notamment sur le classique théorème "Cav(u)" en présentant plusieurs preuves existantes ainsi que des résultats concernant la vitesse de convergence. On définira aussi le jeu dual introduit par De Meyer [**23**], et on mentionnera certains résultats asymptotiques obtenus en utilisant ce jeu ([**32**], [**40**]).

Le deuxième aspect important concerne la vitesse de convergence dans le théorème "Cav(u)" ou de manière équivalente l'étude asymptotique du terme d'erreur dans les jeux répétés à information incomplète d'un côté. On présentera les résultats dus à Mertens et Zamir ([**42**], [**44**]) ainsi qu'à De Meyer ([**23**], [**22**]) et les travaux proprement liés aux problèmes de variation maximale de martingales qui apparaissent dans ces études asymptotiques ([**43**], [**24**]). Ces résultats faisant intervenir un deuxième terme dans le développement asymptotique de la fonction valeur d'un jeu répété seront regroupés abusivement sous le terme de développement du second ordre par opposition au théorème "Cav(u)" qui sera pour nous un résultat du premier ordre. On mentionne dans la famille des résultats du second ordre les jeux financiers introduits par De Meyer [**25**], qui généralisent les précédents résultats de Mertens-Zamir [**43**] dans un cadre où les ensembles d'actions et l'ensemble d'états sont infinis. Ces jeux seront étudiés en détail dans le chapitre 3.

Nous citerons ensuite certains résultats obtenus en théorie des jeux différentiels avec information incomplète d'un côté par Cardaliaguet et Rainer [**17**]. Nous ne rentrerons pas dans les détails de la définition de la valeur ou des stratégies pour ces jeux que nous ne réutiliserons plus par la suite dans cette thèse. Nous présenterons simplement quelques caractérisations, en termes de solutions de viscosité d'équations aux dérivées partielles, des fonctions valeurs de jeux différentiels ainsi que leur représentation probabiliste en termes de problème d'optimisation sur un espace de martingales.

On présentera enfin rapidement les différents chapitres de cette thèse, les principaux résultats obtenus et comment ils s'inscrivent dans le cadre précédemment décrit.

1. Jeux répétés à information incomplète d'un côté.

Nous présentons dans cette section le modèle classique de jeux répétés à information incomplète d'un côté introduit dans Aumann-Maschler [**3**] dans le cas fini (plus précisément avec espaces d'actions et espace d'état finis). Nous nous concentrons principalement sur le résultat asymptotique connu sous le nom de théorème "Cav(u)" et sur l'apparition d'une fonctionnelle appelée L^1-variation d'une martingale. Nous présentons aussi la notion de jeu dual introduite par De Meyer dans [**23**], ainsi que les formules de récurrence associées au jeu initial (appelé jeu primal) et au jeu dual mettant en évidence leur structure récursive.

1.1. Le modèle classique. Un jeu répété à somme nulle à information incomplète d'un côté noté G_n est défini par (I, J, K, A) où I, J, K sont trois ensemble finis et A est une famille de matrices à coefficients réels $A^k = (A^k_{i,j})_{(i,j)\in I\times J}$ indexées par $k \in K$. On considère deux joueurs appelés joueur 1 (J1) et joueur 2 (J2). I et J représentent respectivement les ensembles d'actions de J1 et de J2. K représente l'espace d'état du jeu. A^k représente la fonction de paiement de J1 du jeu dans l'état k.

Dans toute la suite, le cardinal d'un ensemble fini K sera aussi noté K. L'ensemble des probabilités sur un ensemble fini K sera noté $\Delta(K)$ et sera identifié au simplexe canonique dans $\mathbb{R}^K$. On notera $\{e_1, .., e_K\}$ la base canonique de $\mathbb{R}^K$.

Pour tout $p \in \Delta(K)$, on définit un jeu de la manière suivante :

– Etape 0 : L'état $k \in K$ est tiré au sort suivant la probabilité $p \in \Delta(K)$. Le joueur 1 seulement est informé de k.
– Etape 1 : Les deux joueurs choisissent simultanément et indépendamment une action $i_1 \in I$ et $j_1 \in J$. La paire d'actions choisie (i_1, j_1) est annoncée publiquement.
– Etape q ($q = 2, .., n$) : En fonction de leurs observations $(i_1, j_1, .., i_{q-1}, j_{q-1})$ (appelée encore histoire passée du jeu), les joueurs choisissent simultanément et indépendamment une action $i_q \in I$ et $j_q \in J$. La paire d'actions choisie (i_q, j_q) est annoncée publiquement.
– Etape $n + 1$: Le joueur 1 reçoit un paiement égal à

$$\sum_{q=1}^{n} A^k_{i_q, j_q}.$$

Le jeu étant à somme-nulle, J2 reçoit l'opposé de ce paiement.

La description du jeu et en particulier la probabilité initiale sur l'état p est supposée connaissance commune des deux joueurs. En particulier, J2 sait que J1 a été informé de la valeur de la variable d'état k, ne connaît pas cette valeur mais connaît la probabilité initiale sur l'état p qui a été utilisée.

La description ci-dessus correspond au jeu en stratégies pures, considérons maintenant le jeu en stratégies de comportement où à chaque étape $q = 1, .., n$, les joueurs peuvent utiliser une loterie (ou loi de probabilité) pour sélectionner leur action. Ce jeu étendu sera noté $G_n(p)$. Une stratégie de comportement pour J1 est une suite $\sigma = (\sigma_1, .., \sigma_n)$, où pour tout $q = 1, .., n$, σ_q est une probabilité de transition de $K \times I^{q-1} \times J^{q-1}$ dans I. $\sigma_q(k, i_1, j_1, .., i_{q-1}, j_{q-1})[i_q]$ représente la loterie utilisée par J1 pour sélectionner son action i_q si la variable d'état est k et si l'histoire passée du jeu est $(i_1, j_1, .., i_{q-1}, j_{q-1})$. On notera Σ_n l'ensemble des stratégies de comportement de J1. De manière similaire, à la différence près que J2 ne connaît pas la valeur de la variable d'état du jeu, une stratégie de comportement de J2 est une suite $\tau = (\tau_1, .., \tau_n)$, où pour tout $q = 1, .., n$, τ_q est une probabilité de transition de $I^{q-1} \times J^{q-1}$ dans J. On notera $\mathcal{T}_n$ l'ensemble des stratégies de comportement de J2. La donnée du triplet (p, σ, τ) induit de manière unique une probabilité $\Pi(p, \sigma, \tau)$ sur l'ensemble $K \times I^n \times J^n$ en utilisant le théorème de Tulcea (voir [**46**]). Le paiement de J1 dans le jeu $G_n(p)$ est alors

$$\mathbb{E}_{\Pi(p,\sigma,\tau)}[\sum_{q=1}^{n} A^k_{i_q,j_q}] = \sum_{k\in K} p^k \left(\sum_{q=1}^{n} \sigma_q(k, i_1, j_1, .., i_{q-1}, j_{q-1})[i_q] \tau_q(i_1, j_1, .., i_{q-1}, j_{q-1})[j_q] A^k_{i_q,j_q} \right).$$

Notez que l'on prend pour convention que k,i_q,j_q sont les application projections coordonnées sur l'ensemble produit $K \times I^n \times J^n$ et donc considérées comme des variables aléatoires dans l'espérance ci-dessus. Ce jeu est un jeu fini et admet donc une fonction valeur définie sur $\Delta(K)$

et notée $V_n(p)$ qui vérifie

$$\begin{aligned} V_n(p) &= \underset{\sigma\in\Sigma_n}{max}\ \underset{\tau\in\mathcal{T}_n}{min}\ \mathbb{E}_{\Pi(p,\sigma,\tau)}[\sum_{q=1}^{n} A^k_{i_q,j_q}] \\ &= \underset{\tau\in\mathcal{T}_n}{min}\ \underset{\sigma\in\Sigma_n}{max}\ \mathbb{E}_{\Pi(p,\sigma,\tau)}[\sum_{q=1}^{n} A^k_{i_q,j_q}]. \end{aligned}$$

1.2. La martingale des a posteriori. Considérons la probabilité $\Pi(p,\sigma,\tau)$ associée à une paire de stratégies fixée. Notons $(\mathcal{H}_q)_{q=1,..,n}$ la filtration sur $K\times I^n\times J^n$ définie [1] par $\mathcal{H}_q = \sigma(i_1,j_1,..i_{q-1},j_{q-1},i_q)$ ($\mathcal{H}_0$ étant donc la tribu grossière). On peut alors définir une martingale $(p_q)_{q=0,..,n}$ de la manière suivante :

$$\forall q=0,..,n, \qquad p_q = \mathbb{E}_{\Pi(p,\sigma,\tau)}[e_k \mid \mathcal{H}_q], \tag{1.1}$$

où comme plus haut k est vue comme une variable aléatoire à valeurs dans K et où $e_k\in\Delta(K)$ représente la masse de Dirac en k. L'espérance est alors définie en utilisant la structure euclidienne de $\Delta(K)$ vu comme le simplexe canonique dans $\mathbb{R}^K$. Notez que $p_0=p$ et que si l'on définit $p_{n+1}=e_k$, alors le processus prolongé $(p_0,..,p_{n+1})$ est encore une $\mathcal{H}_q$ martingale en posant $\mathcal{H}_{n+1}=\sigma(\mathcal{H}_n,k)$ ayant pour loi finale une loi supportée par les sommets du simplexe. Notez aussi que l'on peut rajouter la variable j_q dans la définition de la tribu $\mathcal{H}_q$ sans modifier le résultat car les variables j_q et k sont par construction conditionnellement indépendantes sachant $\mathcal{H}_q$.

Cette martingale représente les croyances du joueur 2 sur la variable d'état. Cette interprétation suppose que le joueur 2 connaît la stratégie du joueur 1 et révise ses croyances sur l'état en fonction de ses observations en utilisant la règle de Bayes. En effet, une identification simple montre que p_q est la loi conditionnelle de la variable d'état sachant $\mathcal{H}_q$. Introduisons maintenant une fonctionnelle qui sera l'un des objets d'étude principaux de cette thèse. On définit la L^1-variation de la martingale $(p_q)_{q=0,..,n}$ par

$$\mathcal{V}^{L^1}_n((p_q)_{q=0,..,n}) = \mathbb{E}[\sum_{q=1}^{n}\|p_q-p_{q-1}\|_1].$$

Cette fonctionnelle vérifie l'inégalité suivante

LEMME 1.1:

$$\mathcal{V}^{L^1}_n((p_q)_{q=0,..,n}) \le \sum_{k\in K}\sqrt{np^k_0(1-p^k_0)}.$$

1.3. Le jeu dual. Etant donné un jeu $G_n(p)$ comme défini plus haut, nous définissons maintenant le jeu dual $G^*_n(x)$ dépendant d'un paramètre $x\in\mathbb{R}^K$. Ce jeu à somme nulle entre deux joueurs J1 et J2 est défini de la manière suivante.

– Etape 0 : J1 choisit de manière privée la variable d'état du jeu en utilisant une loterie $p\in\Delta(K)$.

1. Nous utilisons la notation courante $\sigma(.)$ pour "tribu engendrée par" en espérant qu'aucune ambiguïté ne naîtra de ce doublon qui reviendra tout au long de cette thèse, la lettre σ étant tout aussi classique pour désigner une stratégie du joueur 1.

– Etape 1 : Les deux joueurs jouent "comme" dans le jeu $G_n(.)$. Cela revient à un jeu en une étape où les deux joueurs choisissent simultanément et indépendamment des stratégies de comportement $\sigma \in \Sigma_n$ et $\tau \in \mathcal{T}_n$.
– Etape 2 : J1 reçoit un paiement

$$\mathbb{E}_{(p,\sigma,\tau)}[\sum_{q=1}^{n} A^k_{i_q,j_q} - x^k].$$

Ce jeu est un jeu fini et admet une fonction valeur notée $W_n(x)$ définie sur $\mathbb{R}^K$ vérifiant :

$$\begin{aligned} W_n(x) &= \max_{p\in\Delta(K),\, \sigma\in\Sigma_n} \min_{\tau\in\mathcal{T}_n} \quad \mathbb{E}_{\Pi(p,\sigma,\tau)}[\sum_{q=1}^{n} A^k_{i_q,j_q} - x^k] \\ &= \min_{\tau\in\mathcal{T}_n} \max_{p\in\Delta(K),\, \sigma\in\Sigma_n} \mathbb{E}_{\Pi(p,\sigma,\tau)}[\sum_{q=1}^{n} A^k_{i_q,j_q} - x^k]. \end{aligned}$$

Les principaux résultats concernant le jeu dual sont résumés dans la proposition suivante. L'utilisation de la dualité convexe et de la transformation de Fenchel (notée f^*) justifient l'utilisation du terme dual [2].

Proposition 1.1: *Pour tous $x \in \mathbb{R}^K$ et $p \in \Delta(K)$,*

$$W_n(x) = (-V_n)^*(-x) \qquad et \qquad V_n(p) = -W^*_n(-p),$$

où la fonction V_n est prolongée sur $\mathbb{R}^K$ par la valeur $-\infty$.

De plus, une stratégie optimale de J2 dans $G^*_n(x)$ avec $-x \in \partial(-V_n)(p)$ est optimale dans $G_n(p)$.

La dernière propriété se généralise aux deux joueurs et relie les stratégies optimales (et aussi ε-optimales) du jeu primal et du jeu dual à travers la relation de dualité liant les fonctions valeurs des jeux correspondants.

La notion de jeu dual a été généralisée à des ensembles d'actions infinis (voir Sorin [**54**]) et un jeu dual dans le cadre d'actions infinis avec une variable d'état réelle a été introduit dans Moussa-Saley [**30**]. Un jeu dual similaire à celui introduit dans [**30**] avec ensembles d'actions infinis et variable d'état dans un espace euclidien sera introduit dans le chapitre 1. Ce jeu dual sera utilisé dans l'analyse des jeux linéaires et dans le modèle de jeux d'échanges financiers au chapitre 3, on introduira aussi une généralisation du même type dans un modèle de jeu à somme non-nulle dans le dernier chapitre de cette thèse.

1.4. Formule de récurrence : Primal et Dual. On entend ici par formule de récurrence une formule reliant la valeur d'un jeu répété de longueur $n + 1$ à la valeur du même jeu de longueur n. Ces formules sont basées sur la structure récursive de ces jeux, qui permet de les décomposer à l'aide d'une nouvelle variable d'état, ici la croyance a posteriori du joueur 2 (pour le jeu primal), en un jeu dont le paiement fait intervenir un paiement d'étape dépendant des actions et de la variable d'état initiale et un paiement de continuation dépendant de la variable d'état actualisée, ici la valeur d'un jeu répété de longueur n.

2. Dans tous les jeux à somme nulle que nous considérerons, le joueur 1 maximise, de manière à conserver la présentation classique. Par ailleurs on utilisera systématiquement la transformation de Fenchel dans le sens convexe.

PROPOSITION 1.2: *Pour tout $p \in \Delta(K)$,*

$$V_{n+1}(p) = \max_{\sigma \in \Sigma_1} \min_{\tau \in \mathcal{T}_1} \mathbb{E}_{\Pi(p,\sigma,\tau)}[A^k_{i_1,j_1} + V_{n-1}(p_1)],$$

où p_1 est défini par (1.1).

Ajoutons que les minimum et maximum commutent dans cette formule. Cette formule permet de construire par induction une stratégie optimale pour J1 qui à l'étape q ne dépend que de q et p_{q-1}. La valeur du jeu dual vérifie elle aussi une formule de récurrence

PROPOSITION 1.3: *Pour tout $x \in \mathbb{R}^K$,*

$$W_{n+1}(x) = \min_{\tau \in \mathcal{T}_1} \max_{i \in I} W_n(x - A_{i,\tau}),$$

où $A_{i,\tau} \in \mathbb{R}^K$ est le vecteur de coordonnées $(\sum_{j \in J} \tau[j] A^k_{i,j})_{k \in K}$.

De la même manière que dans le jeu primal, cette formule permet de construire par induction une stratégie optimale du joueur 2 dans $G^*_n(x)$ qui ne dépend à l'étape q que de q et d'une variable auxiliaire $x_q = x - \sum_{l=1}^{q-1} A_{i_l,\tau_l}$. En combinant ce résultat avec la relation existant entre les stratégies optimales de J2 dans G_n et G^*_n, on obtient un moyen de construire par induction des stratégies optimales de J2 dans G_n. Cette technique sera utilisée dans le chapitre 3 dans le cadre de jeux financiers ainsi que dans le chapitre 4 dans un modèle à somme non-nulle où la structure récursive des équilibres du jeu dual sera par certains aspects similaire.

1.5. Comportement asymptotique : Premier ordre (théorème "Cav(u)"). Introduisons maintenant le jeu non-révélateur, qui est une modification du jeu $G_1(p)$ où le joueur 1 n'a aucune information sur la variable d'état k. Son ensemble de stratégies de comportement est alors un ensemble réduit formé des stratégies σ qui ne dépendent pas de k dans Σ_1, qu'on identifie à $\Delta(I)$. On note $\Pi(\sigma,\tau)$ la loi induite sur $I \times J$. La valeur notée $u(p)$ de ce jeu fini existe et vérifie :

$$u(p) = \max_{\sigma \in \Delta(I)} \min_{\tau \in \Delta(J)} \mathbb{E}_{\Pi(\sigma,\tau)}[\sum_{k \in K} A^k_{i_1,j_1}].$$

On définit la fonction $Cav(u)$ comme étant l'envelope concave de u sur $\Delta(K)$ (la plus petite fonction concave supérieure à u). Le théorème "Cav(u)" s'énonce alors de la manière suivante

THÉOREME 1.1 (Théorème Cav(u)):

$$Cav(u)(p) \leq \frac{1}{n} V_n(p) \leq Cav(u)(p) + C \frac{\sum_{k \in K} \sqrt{p^k(1-p^k)}}{\sqrt{n}}.$$

En particulier, $\frac{1}{n} V_n(p)$ converge vers $Cav(u)(p)$ quand n tend vers $+\infty$.

1.6. Approches duales du théorème "Cav(u)". Mentionnons enfin brièvement deux preuves différentes du théorème "Cav(u)" obtenues par De Meyer et Rosenberg [**32**] et Laraki [**40**] et toutes deux basées sur l'étude du jeu dual. La première repose sur une étude approfondie de la formule de récurrence du jeu dual et sur les propriétés fonctionnelles de la famille d'opérateurs associée (qui transforme V_{n-1} en V_n) et fait notamment apparaître une équation aux dérivées partielles elliptique du type Hamilton-Jacobi. La preuve de Laraki donne un sens précis à cette équation en montrant qu'elle est équivalente à une équation parabolique associée à un jeu différentiel dont la valeur est l'unique solution de viscosité. L'auteur montre alors que le jeu dual est une discrétisation de ce jeu différentiel et en déduit le comportement asymptotique de la valeur du jeu dual. Dans les deux cas, on retrouve le résultat asymptotique énoncé dans le théorème “Cav(u)” grace à la relation de dualité existant entre les fonctions valeurs V_n et W_n.

Les résultats obtenus par Laraki seront généralisés dans la classe des jeux linéaires au chapitre 1.

2. Comportement asymptotique : Deuxième ordre.

Nous présentons dans cette section les résultats de Mertens et Zamir [**42**] et [**43**] pour dans une classe particulière de jeux finis ainsi que le problème de L^1-variation maximale de martingales. Ensuite nous présentons les généralisation obtenues par De Meyer dans [**22**], [**23**] et [**24**].

2.1. La classes de jeux Δ_{σ_0}. Un jeu G_n appartient à la classe Δ_{σ_0} si $I = J$ et si pour tout $p \in \Delta(K)$, J1 a une unique stratégie optimale complètement mixte σ_0 dans le jeu non-révélateur qui ne dépend pas de p. Autrement dit :

$$\forall p \in \Delta(K),\ \ \underset{j\in J}{min} \sum_{(i,k)\in I\times K} p^k A^k_{i,j}\sigma_0[i] \geq u(p).$$

Dans tout jeu de la classe Δ_{σ_0}, la fonction u est linéaire (voir [**22**]). En particulier $u = Cav(u)$, ce qui s'interprète en disant que le joueur 1 ne peut pas guarantir de paiement strictement supérieur à u à une étape du jeu sans utiliser (et potentiellement révéler) en partie son information.

2.2. Le résultat de Mertens et Zamir. Dans le cas particulier où $I = J = K = 2$, Mertens et Zamir ([**42**] ont montré dans un jeu particulier de la classe Δ_{σ_0} que le terme d'erreur dans le théorème “Cav(u)” était d'ordre $\frac{1}{\sqrt{n}}$. Précisément, on a le résultat suivant pour ce jeu.

Théoreme 2.1: *Notons $E_n(p) = (\frac{1}{n}V_n(p) - Cav(u)(p))$, alors pour tout $p \in [0,1]$,*

$$\sqrt{n}E_n \to C\psi(p),$$

où C est une constante et $\psi(x)$ est la densité de la loi normale centrée réduite évaluée à son x-quantile.

La preuve est basée sur le résultat général suivant, démontré par des techniques d'équations différentielles dans [**43**]. Notons $\mathcal{M}_n(p)$ l'ensemble des martingales $(p_0, .., p_n)$ à valeurs dans $[0,1]$ et telles que $p_0 = p \in [0,1]$.

Théoreme 2.2: *Pour tout $p \in [0,1]$*

$$\frac{1}{\sqrt{n}} \underset{(p_0,..,p_n)\in\mathcal{M}_n(p)}{sup} \mathcal{V}_n^{L^1}((p_0, .., p_n)) = \psi(p).$$

Ces résultats sont généralisés à toute la classe Δ_{σ_0} avec $I = J = K = 2$ dans l'article [**44**] où en particulier il est montré qu'en dehors de cette classe de jeux, la vitesse de convergence du terme d'erreur vers 0 est au moins d'ordre $n^{-2/3}$, donc plus rapide.

De plus, une martingale optimale dans ce problème de maximisation à n fixé permet de construire une stratégie optimale de J1 dans le jeu $G_n(p)$. Ce phénomène est en fait très général et valable dans tous les jeux finis et dans la classe plus large des jeux linéaires introduite dans le chapitre 1.

2.3. Les résultats de De Meyer. L'apparition de la loi normale fut expliquée par De Meyer dans [**22**] en utilisant le théorème central limite dans une sous-classe de Δ_{σ_0} pour des ensemble I, J, K finis de cardinal quelconque.

Signalons aussi que ces travaux ont conduit parallèlement a une généralisation du résultat sur la variation L^1-maximale de martingales dans [**24**].

L'extension de ces résultats à la toute la classe Δ_{σ_0} exposée dans De Meyer [**22**] procède différemment. La technique employée est en fait à rapprocher de celle apparaissant dans [**32**] ou dans [**40**]. La preuve donnée par De Meyer étudie les propriétés d'opérateurs associés à la formule de récurrence du jeu dual et utilise une équation aux dérivées partielles du second ordre appelée équation heuristique. Le résultat obtenu est alors un résultat du type "théorème de vérification". Il s'énonce en disant que si l'équation heuristique admet une solution lisse alors $n^{-\frac{1}{2}}W_n(\sqrt{n}x)$ converge uniformément vers cette solution. Une interprétation probabiliste des solutions liée à la loi normale est donnée pour une sous-classe de jeux.

2.4. Modèles de jeux financiers. Les résultats du second ordre présentés ci-dessus trouvent une généralisation naturelle dans le cadre des jeux d'échanges financiers introduits dans De Meyer et Moussa-Saley [**31**] et développés dans De Meyer et Moussa-Saley [**30**], De Meyer et Marino ([**27**] et [**28**]) et plus récemment dans De Meyer [**25**]. Le principal résultat de ce dernier article est l'apparition d'une classe robuste de dynamiques de prix limite, appelée CMMV. En effet, il est montré que ces dynamiques sont les limites des processus de prix à l'équilibre dans une très grande famille de jeux financiers. Les techniques utilisées seront généralisées dans le chapitre 2. D'autre part, ces modèles appartiennent à la classe des jeux linéaires que nous introduirons dans le chapitre 1 et seront étudiés en détails dans le chapitre 3.

3. Quelques résultats concernant les jeux différentiels

Nous présentons ici certains résultats obtenus par Cardaliaguet et Rainer dans [**17**]. Ces résultats font partie d'une littérature plus large sur les jeux différentiels à information incomplète dont différents aspect sont similaires à ceux apparaissant dans l'étude du comportement asymptotique des jeux répétés à information incomplète. Nous renvoyons notamment le lecteur à [**15**] et [**18**] pour des définitions précises des jeux en question, et de la notion de stratégie associée.

3.1. Un jeu différentiel à information incomplète. Les auteurs définissent un jeu différentiel à somme-nulle à information incomplète d'un côté qui apparaît comme la généralisation en temps continu du modèle d'Aumann-Maschler, à la nuance importante près que le problème n'est plus nécéssairement homogène en temps. Ce jeu peut se décrire ainsi. Comme précédemment, le joueur 1 est informé d'une variable d'état $k \in K$ (K fini) tirée au sort selon une loi $p \in \Delta(K)$. Les joueurs ont respectivement des espaces d'actions U, V (compacts de dimension finie). Le jeu se déroule en temps continu dans $[0, 1]$. De manière à faire varier la longueur du jeu, on indexe ce jeu par sa date de début $t_0 \in [0,1]$. Une stratégie open-loop pour J1 est une famille de fonctions mesurables $u^k(s)$ de $[t_0,1]$ dans U (resp. une fonction $v(s)$ de $[t_0,1]$ dans V pour J2). On peut alors définir un paiement

$$J(t_0,p,(u^k(.))_{k\in K},v(.)) = \sum_{k\in K} p^k \int_{t_0}^1 l^k(s,u^k(s),v(s))ds$$

dépendant d'une famille de fonctions l^k continues sur $[0,1]\times U\times V$. La notion qui remplace les stratégies de comportement dans ce cadre est la notion de stratégie mixte non-anticipative avec délai. Notons $\Sigma(t_0)$ (resp. $\mathcal{T}(t_0)$) l'ensemble de telles stratégies pour J1 (resp. J2) dans le jeu commencant en t_0. Un couple $(\alpha,\beta)\in\Sigma(t_0)\times\mathcal{T}(t_0)$ définit une loi de probabilité $\Pi(\alpha,\beta)$ sur les couples de fonctions mesurables (u,v) de $[t_0,1]$ dans $U\times V$. On peut alors définir un jeu sous forme normale où J1 a pour stratégie $(\alpha^k)_{k\in K}\in\Sigma(t_0)^K$ et J2 $\beta\in\mathcal{T}(t_0)$, et dont le paiement est

$$J(t_0,p,\alpha^k(.),\beta(.)) = \sum p^k \mathbb{E}_{\Pi(\alpha^k,\beta)}[\int_{t_0}^1 l^k(s,u^k(s),v(s))ds].$$

Sous la condition dite d'Isaacs suivante,

$$H(t,p) = \underset{u\in U}{max}\ \underset{v\in V}{min} \sum_{k\in K} p^k l^k(u,v) = \underset{v\in V}{min}\ \underset{u\in U}{max} \sum_{k\in K} p^k l^k(u,v),$$

ce jeu a une valeur notée $V(t,p)$ qui est caractérisée de plusieurs manières.

3.2. Représentation probabiliste de la solution et équation associée. Commencons par une représentation faisant intervenir un problème d'optimisation sur un espace de lois de martingales. La fonction valeur V définie plus haut admet la représentation suivante (théorème 3.1 dans [**17**]).

PROPOSITION 3.1:

$$V(t_0,p) = \underset{(X_t)_{t\in[t_0,1]}\in\mathcal{M}(p)}{sup} \mathbb{E}[\int_{t_0}^1 H(s,X_s)ds], \tag{3.1}$$

où $\mathcal{M}(p)$ est l'ensemble des martingales à trajectoires càdlàg sur $[t_0,1]$, à valeur dans $\Delta(K)$ et dont la loi finale X_1 vérifie $\mathbb{P}(X_1=e_k)=p_k$.

Notons que cet ensemble de martingales est la généralisation naturelle en temps continu des martingales de croyances a posteriori introduites plus haut.

Une deuxième représentation de cette fonction est donnée, en terme d'équation aux dérivées partielles avec obstacle. Précisément (proposition 2.5 dans [**17**])

PROPOSITION 3.2: *V est l'unique solution Lipschitz au sens viscosité de l'équation*

$$max\{\frac{\partial\phi}{\partial t}+H(t,p),\quad \lambda_{max}(\frac{\partial^2\phi}{\partial^2 p})\}=0$$

sur $(0,1)\times\Delta(K)$ avec la condition au bord $V(1,p)=0$. En particulier, la fonction $V(t,.)$ est concave par rapport à p.

Cette notion de solution admet aussi une formulation duale equivalente, faisant intervenir la transformée de Fenchel de V par rapport à p (voir [**16**]). Pour ce jeu précis, on a le résultat suivant. La fonction $(t,x) \mapsto (-V(t,.))^*(-x)$ définie sur $\mathbb{R}^K$ est l'unique solution de viscosité de (proposition 3.1. dans [**17**])

$$\begin{cases} \frac{\partial \phi}{\partial t} - H(t, \frac{\partial \phi}{\partial p}) = 0 \\ \phi(1,x) = \underset{k \in K}{sup} \quad (x^k) \end{cases} . \tag{3.2}$$

Dans le cas particulier où les fonctions l^k ne dépendent pas de t, on retrouve alors une caractérisation similaire à celle apparaissant dans les jeux répétés, faisant intervenir l'opérateur de concavification (voir exemple 4.1 dans [**17**]) et la représentation duale donnée dans Laraki [**40**].

L'un des enjeux du chapitre 1 est de démontrer que dans une large classe de jeux contenant les jeux finis, le théorème "Cav(u)", la représentation duale (3.2) ainsi que la représentation probabiliste (3.1) restent valables. On verra ensuite dans le chapitre 2 que la situation est similaire pour les résultats asymptotiques du deuxième ordre dans une sous-classe de jeu où $u = Cav(u) = 0$. En effet, on donnera une représentation duale de la fonction valeur d'un problème asymptotique, faisant intervenir des équations du deuxième ordre et une représentation probabiliste de cette fonction valeur comme problème d'optimisation sur un espace de martingales faisant intervenir un terme de second ordre sous la forme de la variance infinitésimale de ces martingales.

4. Résultats

Nous présentons ici les principaux résultats de cette thèse, en lien avec les résultats généraux en théorie des jeux répétés à information incomplète énoncés précédemment.

4.1. Jeux linéaires. Le premier chapitre de cette thèse concerne une classe de jeux répétés à information complète appelés “Jeux linéaires”. Cette classe contient les jeux finis via une identification très simple, ainsi que leur extension appelée jeu partiellement-révélateur introduite dans [**29**] et utilisée dans [**26**] pour décrire les fonctions valeur d'information. Elle contient aussi la classe des jeux d'échanges financiers introduite dans [**25**] et permet de généraliser ces jeux d'échanges à un cadre multi-actif, ce qui sera traité dans le chapitre 3. Un jeu linéaire est un jeu à information incomplète d'un côté où l'espace d'état est euclidien et où la fonction de paiement du jeu est linéaire par rapport à la variable d'état. Ce premier chapitre a vocation à unifier les différents résultats précédemment présentés en se basant sur la formule de variation de martingale introduite dans [**25**], et sur une généralisation du jeu dual.

On prouve de manière générale que le maxmin d'un jeu linéaire est donné par un problème d'optimisation appartenant à la famille des problèmes de variation de martingales. La norme L^1 qui apparaît dans la notion de L^1-variation est alors remplacée par la fonction valeur du jeu en 1 coup. Ce résultat permet par exemple de généraliser le théorème “Cav(u)” à toute la classe des jeux linéaires de manière directe. On introduit ensuite un jeu dual, et on montre que les résultats de Laraki s'étendent à cette classe de jeu, obtenant ainsi une représentation duale différentielle de $Cav(u)$ sur un simplexe de dimension infinie. On identifie les jeux finis comme une classe particulière de jeux linéaires, et on s'intéresse à leur extension en jeux partiellement révélateurs. Cette extension consiste simplement à considérer le jeu où J1 reçoit seulement un signal aléatoire sur la variable d'état k. Enfin, on prouve que la fonction $Cav(u)$ admet une représentation probabiliste comme la fonction valeur d'un problème d'optimisation sur un espaces de martingales en temps continu. Ce dernier résultat apporte peu d'information sur le comportement asymptotique du jeu, en ce sens que l'ensemble de ses solutions est essentiellement dégénéré, mais il souligne une analogie forte, dans la mesure où les mêmes représentations probabilistes ainsi que les réprésentations duales liées à un problème d'équations aux dérivées partielles apparaissent dans dans le chapitre 2 concernant des développements du second-ordre de fonctions valeurs de jeux linéaires.

4.2. Le problème de variation maximale de martingales. Le but de ce chapitre est de présenter une généralisation des résultats de Mertens-Zamir [**43**] et De Meyer [**25**] sur une famille de problèmes de variation maximale de martingales. Ces problèmes apparaissent de manière très générale dans l'étude des jeux linéaires étudiés au chapitre 1. On s'intéresse ici aux problèmes liés à une classe de jeux possédant une propriété d'invariance spécifique, qui permet de réécrire le problème de variation de martingale associé sous une forme proche de la formulation initiale de la L^1-variation. L'étude de ce problème est intimement liée à l'étude des jeux d'échanges multi-actifs étudiée au chapitre suivant (voir aussi chapitre 1 section 4), modèle dans lequel il apparaît de manière naturelle. Notre étude va néanmoins plus loin et s'applique à une classe de jeux linéaires plus grande que les jeux d'échanges qui seront étudiés au chapitre 3. Les premiers résultats sont la définition d'un problème limite d'optimisation sur un ensemble de lois de martingales en temps continu et la représentation duale de la fonction valeur de ce problème en terme de solutions de viscosité d'une équation aux dérivées partielles du second ordre. Ces deux représentations sont exactement de même nature (mais au deuxième ordre) que les résultats existants pour le premier ordre, dans le cas homogène en temps dans [**17**] et dans Laraki [**40**].

On s'intéresse aussi en détail au comportement asymptotique des lois de martingales de longueur n qui maximisent le problème en temps discret. On montre que les lois limites sont optimales pour le problème limite et on propose une caractérisation de ces lois limites basée sur des résultats de transport optimal et sur les solutions d'un problème de contrôle stochastique lié à l'équation duale. On prouve un théorème de type "vérification" associé à cette caractérisation. On retrouve en particulier les résultats asymptotiques obtenus dans [**25**].

4.3. Applications aux jeux d'échange financiers et aux modèles de prix. Ce chapitre étudie une généralisation multi-actifs du modèle de jeu d'échange introduit dans De Meyer et Moussa-Saley [**31**] puis de manière plus large une sous-classe des jeux linéaires qui généralise les jeux d'échanges introduits dans [**25**]. On étudie une version à deux actifs du jeu introduit dans [**31**] et on prouve l'existence de la valeur de ce jeu et de stratégies optimales pour les deux joueurs. On obtient alors une caractérisation du comportement asymptotique de la valeur du jeu et des processus de prix à l'équilibre en utilisant les résultats du chapitre précédent. On résout alors le problème limite dans plusieurs cas particuliers. Dans un premier temps, on considère un modèle à deux actifs où le deuxième actif est un produit dérivé dont la valeur varie de manière monotone en fonction de l'actif sous-jacent (le premier actif), comme par exemple une option européenne. On montre que dans ce cas, les dynamiques de prix introduites dans [**25**] sont conservées, démontrant ainsi la robustesse de ces dynamiques de prix appelées CMMV face à l'introduction de produits dérivés dans le modèle. On traite aussi un cas particulier où l'hypothèse de monotonie n'est plus respectée, et on construit explicitement une solution du problème limite qui n'est plus dans la classe CMMV, mais est une martingale faisant intervenir le temps local d'un mouvement Brownien.

4.4. Etude asymptotique d'un jeu d'échange à somme non-nulle. Ce chapitre présente l'étude asymptotique d'une classe d'équilibres de Nash dans un modèle de jeu d'échange à somme non-nulle. Ce chapitre est basé sur un travail effectué en collaboration avec Bernard De Meyer.

Le modèle à somme non-nulle exclut la possibilité d'utiliser les techniques de variations de martingales étudiées dans les chapitres précédents.

On aborde le problème en introduisant un jeu dual. On montre que ce jeu dual conserve certaines des propriétés propres au jeu dual dans le cadre à somme nulle. En particulier, on montre l'existence d'une famille particulière d'équilibres ayant une structure récursive. Grace à cette structure, on parvient à obtenir des résultats asymptotiques sur ces équilibres dans le jeu dual. En utilisant la dualité, ces résultats entrainent à leur tour une caractérisation du comportement d'une suite d'équilibres dans le jeu initial. Néanmoins, cette caractérisation est moins directe que la cadre des jeux à somme nulle, car elle est valable dans une suite de jeux où la loi de la variable d'état est remplacée par une approximation. Ceci est dû au fait que la relation existant entre les équilibres du jeu initial et les équilibres du jeu dual est plus complexe que dans le cadre des jeux à somme nulle. Il n'y a plus de fonction valeur, et donc plus de lien entre les variables duales et primales à travers le sous-différentiel de ces fonctions valeurs. Il existe bien un lien de dualité, mais il apparaît de manière implicite, et la principale difficulté dans l'étude de ce modèle tient justement à relier les équilibres du jeu initial à ceux du jeu dual. On prouve dans cette optique un résultat d'existence d'équilibre dans le jeu initial pour une classe de lois régulières, ce qui correspondrait dans le language des jeux à somme-nulle à prouver l'existence d'un élément dans le sous-différentiel de la fonction valeur du jeu primal.

CHAPITRE 1

Jeux linéaires

Ce premier chapitre introduit une classe de jeux répétés à information complète appelés “Jeux linéaires”. On établit dans la première section des résultats généraux et notamment l’extension du théorème “Cav(u)”. Le résultat principal est la formulation de la valeur en tant que problème d’optimisation appelé variation maximale de martingales, résultat que l’on réutilisera au chapitre 3.

Dans la section 2, on définit et on étudie les propriétés du jeu dual, ce qui permet d’étendre les résultats de Laraki [**40**].

On s’intéresse dans la section 3 aux jeux finis et on montre que le problème de variation maximale converge vers un problème limite en temps continu.

Enfin dans la section 4, on identifie la classe des jeux d’échanges financiers comme une sous-classe des jeux linéaires, et on introduit une classe spécifique de jeux motivant le problème étudié dans le chapitre 2.

1. Linear game model

We introduce in this section a general class of repeated games with incomplete information on one side à la Aumann-Maschler [**3**] called linear games. The main result is that the maxmin (or lower value) of these games is equal to the value function of an optimization problem over discrete-time martingale distributions called problem of maximal variation of martingales. This generalizes therefore the former result of De Meyer [**25**] in a multi-dimensional context. Using this representation, we extend the classical “Cav(u)” theorem of Aumann-Maschler for the maxmin functions. The limiting function given by this theorem is then shown to be the value function of an optimization problem over continuous-time martingale distributions and we prove that the limit of any convergent sequence of maximizers for the maximal variation problem is a maximizer of the limit problem.

1.1. The model. We describe now a general two players (P1 and P2) zero-sum game with incomplete information on one side.The state variable is a vector $L = (L^1, .., L^d) \in \mathbb{R}^d$ and is drawn by Nature at the beginning of the game using some probability μ in the set $\Delta^1(\mathbb{R}^d)$ of probabilities with finite first order moment. P1 is informed of the realization of the random variable L while P2 knows only its probability distribution μ. The two players have polish (complete separable metric) action spaces I,J endowed with their Borel σ-fields $\mathcal{I}, \mathcal{J}$. A pure strategy for P1 in the one-shot game is a Borel mapping $s \in \mathcal{L}^0(\mathbb{R}^d, I)$ from $\mathbb{R}^d$ to I, and for P2 some point $j \in J$. T is a bounded Borel mapping from $I \times J$ to $\mathbb{R}^d$. The payoff function in pure strategies for this game is then defined on $\Delta^1(\mathbb{R}^d) \times \mathcal{L}^0(\mathbb{R}^d, I) \times J$ by

$$g(\mu, s, j) = \mathbb{E}_\mu[\langle L, T(s(L), j)\rangle] = \int_{\mathbb{R}^d} \langle x, T(s(x), j)\rangle d\mu(x)$$

Let us denote $\Gamma_n(\mu)$ the associated n-times repeated game in behavioral strategies. At round k ($k = 1, .., n$), P1 and P2 select simultaneously and independently an action $i_k \in I$ for P1 and $j_k \in J$ for P2 using some lottery depending on their information and past observations. Actions are announced publicly after each round. Formally, a behavioral strategy σ for P1 is a sequence $(\sigma_1, ..., \sigma_n)$ of transition probabilities depending on his information and past observations

$$\sigma_k : \mathbb{R}^d \times (I \times J)^{(k-1)} \to \Delta(I)$$

where $\sigma_k(L, i_1, j_1, ..., i_{k-1}, j_{k-1})$ denotes the lottery used to select the action i_k played at round k by P1 when the state variable is L and the past history of the game is $(i_1, j_1, ..., i_{k-1}, j_{k-1})$. Let Σ_n be the set of behavioral strategies for P1. Similarly, a strategy τ for P2 is a sequence $(\tau_1, ..., \tau_n)$ depending only of his past observations

$$\tau_k : (I \times J)^{(k-1)} \to \Delta(J)$$

Let $\mathcal{T}_n$ denote the set of behavioral strategies for P2. A triplet (μ, σ, τ) induces by Tulcea's theorem (see [**46**]) a unique probability $\Pi_{(\mu,\sigma,\tau)} \in \Delta(\mathbb{R}^d \times I^n \times J^n)$. The payoff function in $\Gamma_n(\mu)$ is given by

$$g_n(\mu, \sigma, \tau) = \mathbb{E}_{\Pi(\mu,\sigma,\tau)}[\langle L, \sum_{k=1}^{n} T(i_k, j_k)\rangle]$$

Note that the function inside the above expectation is a linear function of the state variable L which justifies the name of linear games for this class.

1.2. The martingale of expected beliefs. As usual, if P2 knows that P1 is using some strategy σ, he will update his beliefs on the state variable L given his observations using Bayes' rules. In a more general context, beliefs of P2 should be represented by the conditional distribution of the state variable over $\mathbb{R}^d$ given his information. In our case, due to the linearity of the payoffs, we shall only consider the process of expected beliefs of P2. Precisely, let us define the expected belief of P2 as the expected value of L given his information. Available information after round k is represented by the σ-field $\mathcal{F}_k = \sigma(i_1, j_1, ..., i_k, j_k)$ of past observations. Formally, the process of expected beliefs is the martingale

$$L_k = \mathbb{E}_{\Pi(\mu,\sigma,\tau)}[L \mid \mathcal{F}_k] \tag{1.1}$$

and the law of this martingale depends on the pair of strategies (σ, τ). This martingale has length n and its final value L_n is by construction Blackwell dominated [1] by μ. The initial value is by convention $L_0 = \mathbb{E}_\mu[L]$ and we will sometimes consider the variable $L_{n+1} \triangleq L$ as the terminal value of the martingale [2], which follows the law μ.

1.3. The maxmin and minmax functions. Let us define the maxmin and minmax of the game $\Gamma_n(\mu)$, respectively the maximal and minimal payoffs P1 and P2 can guarantee:

$$\underline{V}_n(\mu) = \underset{\sigma\in\Sigma_n}{sup}\ \underset{\tau\in\mathcal{T}_n}{inf}\ \mathbb{E}_{\Pi_{(\mu,\sigma,\tau)}}[\langle L, \sum_{k=1}^n T(i_k, j_k)\rangle]$$

$$\overline{V}_n(\mu) = \underset{\tau\in\mathcal{T}_n}{inf}\ \underset{\sigma\in\Sigma_n}{sup}\ \mathbb{E}_{\Pi_{(\mu,\sigma,\tau)}}[\langle L, \sum_{k=1}^n T(i_k, j_k)\rangle]$$

NOTATION 1: *In the sequel,* $|.|$ *refers to the usual euclidian norm on* $\mathbb{R}^d$ *and* $|.|_1, |.|_\infty$ *to the classical norm of order* 1 *and uniform norm.*

The following result is classical in zero-sum games.

PROPOSITION 1.1: *The functions* $\underline{V}_n$ *and* $\overline{V}_n$ *are concave on* $\Delta^1(\mathbb{R}^d)$*, nondecreasing for the Blackwell order, Lipschitz continuous with respect to the Wasserstein distance of order 1 (denoted* d_{W_1}*) and positively 1-homogenous in the following sense*

$$\forall \lambda > 0, \quad \underline{V}_n([\lambda Y]) = \lambda \underline{V}_n([Y]) \tag{1.2}$$

where Y *is some integrable random variable and* $[Y]$ *denotes the law of* Y*.*

PROOF. Let us prove the nondecreasing property. Suppose that $\mu_1 \preceq \mu_2$, and let (Y_1, Y_2) be a martingale such that $Y_i \sim \mu_i$ for $i = 1, 2$. Let σ be a strategy in $\Gamma_1(\mu_1)$. Define the variable i_1 such that its conditional law given (Y_1, Y_2) is $\sigma(Y_1)$. The law of (Y_2, i_1) defines a strategy for

1. Recall that μ_1 is Blackwell-dominated by μ_2 ($\mu_1 \preceq \mu_2$) if there exists a two-steps martingale X_1, X_2 such that X_i is μ_i distributed for $i = 1, 2$.

2. Note that it is always possible to add an additional variable having law μ to a martingale whose final distribution is Blackwell dominated by μ

P1 in $\Gamma_1(\mu_2)$. Using the martingale property, the induced payoff in $\Gamma_1(\mu_2)$ against any strategy τ is

$$\mathbb{E}[\langle Y_2, T(i_1, j_1)\rangle] = \mathbb{E}[\langle Y_1, T(i_1, j_1)\rangle] = g(\mu_1, \sigma, \tau)$$

implying that P1 can guarantee at least the same quantity in $\Gamma_1(\mu_2)$ than in $\Gamma_1(\mu_1)$. For the Lipschitz property, let $\mu_1, \mu_2 \in \Delta^1(\mathbb{R}^d)$ and (Y_1, Y_2) be random variables such that $Y_i \sim \mu_i$ for $i = 1, 2$ and $\mathbb{E}[|Y_1 - Y_2|_1] = d_{W_1}(\mu_1, \mu_2)$. Let $(i_1, j_1) \in I \times J$ be any random variables defined on the same probability space, the result follows easily from the following inequality

$$|\mathbb{E}[\langle Y_1 - Y_2, T(i_1, j_1)\rangle]| \leq C_T d_{W_1}(\mu_1, \mu_2)$$

where $C_T = sup_{(i,j)\in I\times J}|T(i,j)|_\infty$. For the concavity, let $\mu_1, \mu_2 \in \Delta^1(\mathbb{R}^d)$ and $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$ with $\lambda \in (0, 1)$. There exists a joint law of the pair (i, L) on $\{1, 2\} \times \mathbb{R}^d$ such that $\mathbb{P}(i = 1) = \lambda$ and the conditional law of L given i is $(\mu_i)_{i=1,2}$. In the game $\Gamma_1(\mu)$, P1 can select i using the conditional law of i given L by using some exogenous lottery and then play an ε-optimally in the game $\Gamma_1(\mu_i)$. This process defines a strategy in $\Gamma_1(\mu)$ which guarantees $\lambda\underline{V}_1(\mu_1) + (1 - \lambda)\underline{V}_1(\mu_1) - \varepsilon$. The same arguments work for $\underline{V}_n$ and $\overline{V}_n$. □

1.4. The maximal variation problem. Let us introduce some notations

Definition 1.1: *For $\mu \in \Delta^1(\mathbb{R}^d)$*

- $\mathfrak{M}_n(\mu)$ *is the collection of martingales $(L_k, \mathcal{F}_k)_{k=1,..,n}$ defined on some filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_k)_{k=0,..,n}, \mathbb{P})$, of length n and whose final distribution $[L_n]$ is Blackwell dominated by μ. By convention, we set $\mathcal{F}_0 = \{\Omega, \emptyset\}$.*
- $\mathcal{M}_n(\mu)$ *is the subset of $\Delta((\mathbb{R}^d)^n)$ formed by the laws of $\mathbb{R}^d$-valued martingales $(L_1, .., L_n)$ such that the final distribution $[L_n]$ is Blackwell dominated by μ.*

Definition 1.2: *Given a function $F : \Delta^1(\mathbb{R}^d) \to \mathbb{R}$, we define the F-variation on the set $\mathfrak{M}_n(\mu)$ by*

$$\mathcal{V}_n^F((L_k, \mathcal{F}_k)_{k=1,..,n}) = \mathbb{E}[\sum_{k=1}^n F([L_k \mid \mathcal{F}_{k-1}])] \tag{1.3}$$

where $[L_k \mid \mathcal{F}_{k-1}]$ denotes the conditional law of L_k given $\mathcal{F}_{k-1}$.
The maximal F-variation is then defined as

$$\overline{\mathcal{V}}_n^F(\mu) = \sup_{\mathfrak{M}_n(\mu)} \mathcal{V}_n^F.$$

In the sequel, when $F = \underline{V}_1$, we will use the shorter notations $\mathcal{V}_n \triangleq \mathcal{V}_n^{\underline{V}_1}$ and $\overline{\mathcal{V}}_n \triangleq \overline{\mathcal{V}}_n^{\underline{V}_1}$.

The following property results directly from the concavity of $\underline{V}_1$.

Lemma 1.1: *For all $\mu \in \Delta^1(\mathbb{R}^d)$, we have*

$$\overline{\mathcal{V}}_n(\mu) = \sup_{[(L_k)_{k=1,..,n}]\in\mathcal{M}_n(\mu)} \mathcal{V}_n((L_k, \mathcal{F}_k^L)_{k=1,..,n}) \tag{1.4}$$

if we define $(\mathcal{F}_k^L)_{k=0,1,..,n}$ as the natural filtration of $(L_1, .., L_n)$, i.e.

$$\mathcal{F}_k^L = \sigma(L_1, .., L_k) \text{ for } k = 1, .., n \text{ and } \mathcal{F}_0 = \{(\mathbb{R}^d)^n, \emptyset\}$$

PROOF. It is clearly sufficient to prove that $\overline{\mathcal{V}}_n(\mu)$ is not greater than the right-hand side of (1.4). Let $((L_k,\mathcal{F}_k)_{k=1,..,n}) \in \mathfrak{M}_n(\mu)$. Since $\underline{V}_1$ is concave and d_{W_1}-Lipschitz, it follows from Jensen's inequality (see lemma 5.1) that for all $k=1,..,n$

$$\underline{V}_1([L_k \mid \mathcal{F}_{k-1}]) \leq \underline{V}_1([L_k \mid \mathcal{F}^L_{k-1}])$$

The proof follows then by summation over k. □

Let us now prove that the maxmin of the n-rounds game is given by the maximal $\underline{V}_1$-variation problem.

PROPOSITION 1.2:

$$\underline{V}_n(\mu) = \overline{\mathcal{V}}_n(\mu)$$

Moreover, any ε-maximizer of $\overline{\mathcal{V}}_n$ induces an (2ε)-optimal strategy for P1.

We will need the following two lemmas that are essentially classical measurable selection results.

LEMMA 1.2: *For all $\varepsilon > 0$, there exists a measurable function*

$$\varphi_\varepsilon : \mathbb{R}^d \times \Delta^1(\mathbb{R}^d) \times [0,1] \to I$$

such that the strategy induced by $i_1 = \varphi_\varepsilon(L,\mu,U)$ where U is some uniform random variable on $[0,1]$ independent from L is ε-optimal in the game $\Gamma_1(\mu)$ for all $\mu \in \Delta^1(\mathbb{R}^d)$.

The proof of this first lemma is standard but quite long and therefore postponed to section 5. Let us denote $\Delta^1(I\times\mathbb{R}^d)$ the set of joint probability distributions on $I\times\mathbb{R}^d$ such that the marginal probability induced on $\mathbb{R}^d$ is in $\Delta^1(\mathbb{R}^d)$. Note that a pair (σ_1,μ) where σ_1 is a strategy of P1 in the game $\Gamma_1(\mu)$ defines naturally a probability in $\Delta^1(I\times\mathbb{R}^d)$ that will be denoted $\pi(\sigma_1,\mu)$.

LEMMA 1.3: *For all $\varepsilon > 0$, there exists a universally measurable function*

$$\tau_\varepsilon : \Delta^1(I\times\mathbb{R}^d) \to \Delta(J)$$

such that for all (σ_1,μ)

$$\mathbb{E}_{\Pi(\mu,\sigma_1,\tau_\varepsilon(\pi(\sigma_1,\mu)))}[\langle L, T(i_1,j_1)\rangle] \leq \underline{V}_1(\mu)+\varepsilon$$

PROOF. Endow the set $\Delta^1(I\times\mathbb{R}^d)$ with the coarsest topology such that the applications $\pi \to \int g(i,x)d\pi(i,x)$ are continuous for all real-valued continuous functions g such that $|g(i,x)| \leq C(1+|x|)$ for some constant C independent of i. If we endow $\Delta(J)$ with the usual weak topology, then the application

$$(\pi,\tau) \to \int \langle x, T(i,j)\rangle d\pi(i,x)\otimes\tau(j)$$

is jointly measurable. If μ denotes the marginal law on $\mathbb{R}^d$ induced by π, the set

$$\{(\pi,\tau) \,:\, \int \langle x, T(i,j)\rangle d\pi(i,x)\otimes\tau(j) \leq \underline{V}_1(\mu)+\varepsilon\}$$

is therefore a Borel subset of $\Delta^1(I\times\mathbb{R}^d)\times\Delta(J)$. The existence of an ε-optimal universally measurable selection ψ_ϵ follows therefore from Von Neumann's selection theorem (see appendix theorem 2.3). □

Proof of proposition 1.2. This proof is just an adaptation of the one appearing in [**25**] that we tried to make as precise as possible for the purpose of this work. Note however that the second part is slightly different since we consider the maxmin of the game and not the value. Let us start with an ε-maximizer $(L_k)_{k=1,..,n}$ for the problem $\overline{\mathcal{V}}_n$. Formally, since only the law of the chosen martingale is relevant, given the state variable L and a sequence $U_1, .., U_n$ of independent random variables uniformly distributed on $[0,1]$ and independent from L, there exists a sequence of measurable functions (f_1, f_n) such that the martingale

$$L_k = f_k(L, U_1, .., U_k) \text{ for } k = 1, ..n$$

has the same law as the initially chosen martingale (so that we use the same notation). Now from the previous lemma, we can define P1's strategy σ as follows: given a sequence $(Y_1, .., Y_n)$ of independent random variables uniformly distributed on $[0,1]$ and independent from $(L, U_1, .., U_n)$, the action i_k of P1 at stage k is

$$i_k(L_1, .., L_k, Y_k) = \varphi_\varepsilon(L_k, [L_k \mid L_1, .., L_{k-1}], Y_k)$$

This does not define directly a behavioral strategy, since P1 has to remember at each stage the value of the auxiliary variables L_k (or U_k). But this mixed strategy defines a joint law on $(L, i_1, .., i_n)$ which can always be disintegrated in a behavioral strategy that does not depend on P2's actions [3]. We can clearly keep the above representation of P1's strategy to compute his payoff against some strategy τ, even if it is not expressed as a behavioral strategy, since these computations depend only on the induced law on $(L, i_1, ..i_n)$. Without loss of generality, given a sequence $(W_1, .., W_n)$ of independent random variables uniformly distributed on $[0,1]$ and independent from $(L, U_1, Y_1, .., U_n, Y_n)$ we can assume that the strategy τ is given by a sequence $g_1, .., g_n$ of measurable functions such that the action of P2 at stage k is given by

$$j_k = g_k(i_1, j_1, .., i_{k-1}, j_{k-1}, W_k)$$

Let us define the filtration $(\mathcal{H}_k)_{k=1,..,n}$ by $\mathcal{H}_k = \sigma(L_1, Y_1, W_1, .., L_k, Y_k, W_k)$. With this notation, variables i_k, j_k are $\mathcal{H}_k$-measurable and we have the following equalities between the conditional laws for all $k = 1, .., n$

$$[L \mid \mathcal{H}_k] = [L \mid L_1, .., L_k], \quad [L_k \mid \mathcal{H}_{k-1}] = [L_k \mid L_1, .., L_{k-1}]$$

The conditional payoff at round k given the past actions of the players is

$$\begin{aligned}\mathbb{E}[\langle L, T(i_k, j_k)\rangle \mid i_1, j_1, .., i_{k-1}, j_{k-1}] &= \mathbb{E}[\mathbb{E}[\langle L, T(i_k, j_k)\rangle \mid \mathcal{H}_k] \mid i_1, j_1, .., i_{k-1}, j_{k-1}] \\ = \mathbb{E}[\langle L_k, T(i_k, j_k) \mid i_1, j_1, .., i_{k-1}, j_{k-1}] &= \mathbb{E}[\mathbb{E}[\langle L_k, T(i_k, j_k)\rangle \mid \mathcal{H}_{k-1}] \mid i_1, j_1, .., i_{k-1}, j_{k-1}]\end{aligned}$$

where the second equality follows from the linearity of the payoff. Finally the conditional expectation given $\mathcal{H}_{k-1}$ above is exactly the payoff in a one-round game where the law of the state variable is $[L_k \mid L_1, .., L_{k-1}]$ and the joint conditional law of $[L_k, i_k \mid L_1, .., L_{k-1}]$ has been constructed so that

$$\mathbb{E}[\langle L_k, T(i_k, j_k)\rangle \mid \mathcal{H}_{k-1}] \geq \underline{V}_1([L_k \mid L_1, .., L_{k-1}]) - \varepsilon$$

3. There is no need to refer to the general Kuhn's theorem here (which applies however in our case), since this strategy is actually a strategy in a 1 player game.

Summing up these inequalities proves that $V_n(\mu) \geq \overline{\mathcal{V}}_n - n\varepsilon$ and we obtain a first inequality by sending ε to zero.

It remains to prove the reverse inequality. Let us fix a pair (μ, σ) where σ is a behavioral strategy for P1 in $\Gamma_n(\mu)$ and some $\varepsilon > 0$. We will construct a strategy τ for P2 by induction such that for all $k = 1, .., n$ the expected payoff at round k is not greater than

$$\mathbb{E}_{\Pi(\mu,\sigma,\tau)}[\underline{V}_1([L_k \mid i_1, j_1, .., i_{k-1}, j_{k-1}])] + \varepsilon$$

where L_k is defined by $\mathbb{E}_{\Pi(\mu,\sigma,\tau)}[L \mid i_1, j_1, .., i_{k-1}, j_{k-1}, i_k]$. Note for this that adding j_k in the conditional expectation defining L_k does not change its value, which does not depend on τ_k. Suppose that $(\tau_1, .., \tau_{k-1})$ is already constructed. Since $(\mu, \sigma, \tau_1, .., \tau_{k-1})$ defines a joint law on $(L, i_1, .., i_k, j_1, .., j_{k-1})$ that will be denoted $\mathbb{P}$, the conditional expectation L_k is well-defined and $\mathbb{P}$-almost surely equal to a Borel mapping $f_k(i_k, i_1, j_1, ..., i_{k-1}, j_{k-1})$. The disintegration theorem (appendix theorem 1.1) implies the existence of a $\Delta^1(\mathbb{R}^d \times I)$-valued Borel mapping $M_k(i_1, j_1, ..., i_{k-1}, j_{k-1})$ which is a regular version of the conditional law $[L_k, i_k \mid i_1, j_1, ..., i_{k-1}, j_{k-1}]$. Using lemma 1.3, the mapping τ_ε being universally measurable, there exists a Borel mapping $\tilde{\tau}_\varepsilon$ which is almost surely equal to τ_ε with respect to the law of the random variable $[L_k, i_k \mid i_1, j_1, ..., i_{k-1}, j_{k-1}]$ (or the image probability of $\mathbb{P}$ induced by the mapping M_k which therefore depends of $\mathbb{P}$ which is known by P2). The strategy $\tau_k = \tilde{\tau}_\varepsilon(M_k(i_1, j_1, ..., i_{k-1}, j_{k-1}))$ has then the required properties. The overall payoff of P2 is therefore not greater than

$$\mathbb{E}_{\Pi(\mu,\sigma,\tau)}[\sum_{k=1}^{n} \underline{V}_1([L_k \mid i_1, j_1, .., i_{k-1}, j_{k-1}])] + n\varepsilon \leq \overline{\mathcal{V}}_n(\mu) + n\varepsilon$$

which concludes the proof since ε was arbitrary. □

Another useful property is the following, which is theorem 4 in [**25**] and is only reproduced in section 5 for the sake of completeness since the proof is exactly the same.

Proposition 1.3: *Assume that for all $\mu \in \Delta^1(\mathbb{R}^d)$, the game $\Gamma_n(\mu)$ has a value (i.e. $\underline{V}_n(\mu) = \overline{V}_n(\mu)$. If (σ, τ) is a pair of optimal strategies in $\Gamma_n(\mu)$, then the expected belief martingale induced by (μ, σ, τ) is optimal in the maximization problem $\overline{\mathcal{V}}_n(\mu)$.*

Remark 1.1: *The proof of proposition 1.2 implies that if a strategy of P1 is ε-optimal in $\Gamma_n(\mu)$ and does not depend on the actions of P2, then the induced martingale of expected beliefs is ε-optimal for $\overline{\mathcal{V}_n}(\mu)$ independently of the strategy of P2.*

The non-revealing game. Let us now define the maxmin and minmax of the non-revealing game, which is a modified version of $\Gamma_1(\mu)$ in which P1 is not informed of the realization L. In the following definition, the set of strategies in Σ_1 that do not depend on L is identified with $\Delta(I)$.

$$\underline{u}(\mu) \triangleq \underset{\sigma\in\Delta(I)}{sup}\ \underset{\tau\in\Delta(J)}{inf}\ \mathbb{E}_{\Pi_{(\mu,\sigma,\tau)}}[\langle L, T(i_1, j_1)\rangle]$$

$$\overline{u}(\mu) \triangleq \underset{\tau\in\Delta(J)}{inf}\ \underset{\sigma\in\Delta(I)}{sup}\ \mathbb{E}_{\Pi_{(\mu,\sigma,\tau)}}[\langle L, T(i_1, j_1)\rangle]$$

Let us list some easy properties

LEMMA 1.4:
$$\forall \mu \in \Delta^1(\mathbb{R}^d),\ \underline{u}(\mu) = \underline{V}_1(\delta_{\mathbb{E}(\mu)}) \text{ and } \overline{u}(\mu) = \overline{V}_1(\delta_{\mathbb{E}(\mu)})$$
with $\mathbb{E}(\mu) = \int_{\mathbb{R}^d} x d\mu(x) \in \mathbb{R}^d$ *and* δ_x *denotes the Dirac mass at* x.

The functions $\overline{u}$ *and* $\underline{u}$ *are Lipschitz with respect to the distance* d_{W_1} *and positively homogenous in the sense of* (1.2).

PROOF. The equality relating $\underline{u}$ to $\underline{V}_1$ follows easily from the independence of (i_1, j_1) and L, which imply that we can integrate the payoff with respect to L at first by Fubini's theorem. The other properties can be proven as in lemma 1.1. □

1.5. Reduced strategies. We introduce the notion of reduced strategies, which are simply strategies that do not depend on P2's past actions.
A reduced strategy τ for P2 is a sequence $(\tau_1, ..., \tau_n)$ of transition probabilities
$$\tau_k : (I)^{k-1} \to \Delta(J).$$
A reduced strategy for P1 is a sequence $\sigma = (\sigma_1, ..., \sigma_n)$ of transition probabilities
$$\sigma_k : \mathbb{R}^d \times (I)^{k-1} \to \Delta(J).$$
Let Σ_n^r (resp. $\mathcal{T}_n^r$) the set of reduced behavioral strategies of P1 (resp. P2). Given a reduced strategy σ of P1, the pair (μ, σ) induces a probability on $\mathbb{R}^d \times (I)^n$. The next proposition implies that an optimal strategy in the game restricted to reduced strategies is still optimal in the initial game.

PROPOSITION 1.4: *If a strategy guarantees the quantity* C *in the game* $\Gamma_n(\mu)$ *where the strategy sets of the players are restricted to reduced strategies, then it guarantees* C *in the initial game.*

PROOF. This proof is a direct adaptation of the one appearing in [**23**]. For any $\tau \in \mathcal{T}_n$, there exists a reduced strategy $\widehat{\tau}$ giving the same payoff as τ against any reduced strategy of P1. The strategy $\widehat{\tau}$ proceeds as follows : at step k, P2 does not remind his past actions $(j_1, .., j_{k-1})$, but using past actions of P1, he generates a virtual history $(i_q, \widehat{j_q})_{q=1,..,k-1}$ by choosing $\widehat{j_q}$ with the probability $\tau(i_1, \widehat{j_1}, .., i_{q-1}, \widehat{j_{q-1}})$. He selects then at stage k an action j_k with the probability $\tau((i_q, \widehat{j_q})_{q=1,..,k-1})$. Since the action of P1 does not depend on past actions of P2, the conditional distribution of (i_k, j_k) given $(L, i_1, .., i_{k-1})$ is the same as if P2 was using τ, and so is the conditional expected payoff at stage k. The situation is not symmetric for P1, because to generate a virtual history of the past actions of P2, he has to know which strategy P2 is using. However, given $\tau^* \in \mathcal{T}_n^r$, the same argument shows that for all $\sigma \in \Sigma_n$, there exists a reduced strategy $\widehat{\sigma}$ giving the same payoff as σ against the fixed strategy τ^*, which allows to conclude. □

1.6. The "Cav(u)" theorem. We assume here and in the following section that $\mu \in \Delta(P) \subset \Delta^1(\mathbb{R}^d)$ for some compact convex subset P of $\mathbb{R}^d$. We are now able to state the classical "Cav(u)" theorem of Aumann-Maschler in this context whose proof is divided in two propositions.

Proposition 1.5: *For all μ in $\Delta(P)$, we have*

$$Cav(\underline{u})(\mu) \leq \frac{1}{n}\underline{V}_n(\mu) \leq Cav(\underline{u})(\mu) + \frac{1}{n}\overline{\mathcal{V}}_n^{N_1}(\mu)$$

where N_1 is the centered L_1 norm defined on the set $\Delta^1(\mathbb{R}^d)$ by $N_1(\mu) = \mathbb{E}[|X - \mathbb{E}[X]|_1]$ for $X \sim \mu$ and Cav denotes the concavification operator on $\Delta(P)$.

Proof. The proof proceeds exactly as in the finite case (see [**41**] propositions 5.2.8-9). Let us fix $\varepsilon > 0$. At first, let $(\lambda_q)_{q=1,..,l}$ be a finite convex combination and $(\mu_q)_{q=1,..,l}$ in $\Delta(P)$ such that

$$\sum_{q=1}^{l} \lambda_q \underline{u}(\mu_q) \geq Cav(\underline{u})(\mu) - \varepsilon$$

There exists a joint law of the pair (r, L) on $\{1, .., l\} \times \mathbb{R}^d$ such that $\mathbb{P}(r = q) = \lambda_q$ and the conditional law of L given r is $(\mu_r)_{r=1,..,l}$. P1 selects r using the conditional law of r given L using some exogenous lottery and then plays ε-optimally in the game $\Gamma_1(\delta_{\mathbb{E}(\mu_r)})$ at each round (using independent copies of these lotteries and without using any information on L). This process defines a behavioral strategy and this strategy guarantees $nCav(\underline{u}) - 2n\varepsilon$. The first inequality follows by sending ε to zero.

The second inequality follows directly from proposition 1.1. Using the Lipschitz property of $\underline{V}_1$ and the fact that $d_{W_1}(\mu, \delta_{\mathbb{E}(\mu)}) = N_1(\mu)$, we have

$$\forall \mu \in \Delta^1(\mathbb{R}^d), \qquad \underline{V}_1(\mu) \leq \underline{u}(\mu) + C_T N_1(\mu)$$

Applying this inequality in the expression of $\mathcal{V}_n^{\underline{V}_1}((L_k)_{k=1,..,n})$ implies with $L_{n+1} \sim \mu$ a terminal variable added to the martingale.

$$\begin{aligned}
\mathcal{V}_n^{\underline{V}_1}((L_k)_{k=1,..,n}) &\leq \mathbb{E}[\sum_{k=1}^{n} \underline{u}([L_k \mid L_1, .., L_{k-1}])] + C_T \mathcal{V}_n^{N_1}((L_k)_{k=1,..,n}) \\
&\leq \mathbb{E}[\sum_{k=1}^{n} \underline{u}([L_{n+1} \mid L_1, .., L_{k-1}])] + C_T \mathcal{V}_n^{N_1}((L_k)_{k=1,..,n}) \\
&\leq \mathbb{E}[\sum_{k=1}^{n} Cav(\underline{u})([L_{n+1} \mid L_1, .., L_{k-1}])] + C_T \mathcal{V}_n^{N_1}((L_k)_{k=1,..,n}) \\
&\leq \sum_{k=1}^{n} Cav(\underline{u})([L_{n+1}]) + C_T \mathcal{V}_n^{N_1}((L_k)_{k=1,..,n}) \\
&\leq nCav(\underline{u})(\mu) + C_T \mathcal{V}_n^{N_1}((L_k)_{k=1,..,n})
\end{aligned}$$

The second line follows from the definition of $\underline{u}$, the third from the definition of the concavification operator. The fourth one follows from Jensen's inequality (lemma 5.1). Application of Jensen' inequality is possible since $\underline{u}$ is weakly continuous on the weakly compact set $\Delta(P)$ and bounded, which implies that $Cav(\underline{u})$ is weakly upper semi-continuous and bounded (see lemma 26.13 in [**20**]). □

As for the proof of the "Cav(u)" theorem given in [**41**], the error term appearing in the previous result can be easily bounded using classical probability results.

PROPOSITION 1.6: *For all μ in $\Delta(P)$,*

$$\frac{1}{n}\overline{\mathcal{V}}_n^{N_1}(\mu)\underset{n\to\infty}{\longrightarrow}0$$

PROOF. We will actually prove a more precise result, namely that the maximal N_1-variation is bounded by $C\sqrt{n}$ for some constant C depending on μ. Note at first that the centered L_1-norm is given by the following optimization problem

$$N_1(\mu) = \mathbb{E}_\mu[|X-\mathbb{E}[X]|_1] = \underset{[X]=\mu,\, Y\in\{-1,1\}^d}{max}\mathbb{E}[\langle(X-\mathbb{E}[X]), Y\rangle] \tag{1.5}$$

where the maximum is taken over all joint distributions of pairs of random variables (X,Y) fulfilling the mentioned marginal constraints. The maximum is reached for the sign function $Y = S(X-\mathbb{E}[X]) \triangleq (sgn(X_i-\mathbb{E}[X_i])_{i=1,..,d})$. Let us now consider a martingale $(L_1,..,L_{n+1})$ such that $L_{n+1}\sim\mu$. We have

$$\mathcal{V}_n^{N_1}((L_k)_{k=1,..,n}) = \mathbb{E}[\sum_{k=1}^n |L_k-L_{k-1}|_1]$$

Define $Y_k = S(L_k-L_{k-1})$ and $Z_k = Y_k - \mathbb{E}[Y_k \mid \mathcal{F}^L_{k-1}]$. Using the martingale property, we find

$$\begin{aligned}\mathcal{V}_n^{N_1}((L_k)_{k=1,..,n}) &= \mathbb{E}[\sum_{k=1}^n\langle L_k-L_{k-1}, Y_k\rangle] = \mathbb{E}[\sum_{k=1}^n\langle L_k-L_{k-1}, Z_k\rangle]\\ &= \mathbb{E}[\sum_{k=1}^n\langle L_k, Z_k\rangle] = \mathbb{E}[\sum_{k=1}^n\langle L_{n+1}, Z_k\rangle]\\ &= \mathbb{E}[\langle L_{n+1}, \sum_{k=1}^n Z_k\rangle]\end{aligned}$$

Applying Cauchy-Schwartz inequality, we find

$$\frac{1}{\sqrt{n}}\mathcal{V}_n^{N_1}((L_k)_{k=1,..,n}) \le \|L_{n+1}\|_{L^2}\|\frac{1}{\sqrt{n}}\sum_{k=1}^n Z_k\|_{L^2}$$

and the L^2-norm above is bounded by $2\sqrt{d}$ using the orthogonality of the increments of a martingale in L^2 which concludes the proof. □

We can now state the announced extension of the "Cav(u)" theorem.

THEOREM 1.1: *For all $\mu\in\Delta(P)$*

$$\frac{1}{n}\underline{V}_n(\mu)\underset{n\to\infty}{\longrightarrow}Cav(\underline{u})(\mu)$$

REMARK 1.2: *The preceding proof can be extended to $\Delta^p(\mathbb{R}^d)$ for $p>1$ by replacing Cauchy-Schwarz inequality by Holder's inequality and applying then Burkholder's inequality to bound the last term.*

1.7. The limit continuous-time optimization problem. The following continuous-time probabilistic formulation is very easy to obtain from the definition of the concavification operator and is reminiscent from the result given in theorem 3.1 of [**17**] for differential games with incomplete information (possibly non-homogenous in time).

PROPOSITION 1.7: *For all $\mu \in \Delta(P)$,*

$$Cav(\underline{u})(\mu) = \sup_{(X_t)_{t\in[0,1]}\in\mathcal{M}(\preceq\mu)} \mathbb{E}[\int_0^1 g_{\underline{u}}(X_t)dt] \tag{1.6}$$

where $\mathcal{M}(\preceq \mu)$ is the set of (laws of) càdlàg martingales $(X_t)_{t\in[0,1]}$ whose final distribution $[X_1]$ is Blackwell dominated by μ. Moreover, X is a maximizer if and only if for dt-almost all s in $[0,1]$

$$\mathbb{E}[g_{\underline{u}}(X_s)] = Cav(\underline{u})(\mu)$$

PROOF. Note at first that any martingale in $\mathcal{M}(\preceq \mu)$ has trajectories in the set P with probability 1. Since X is a martingale we have for all $s \leq t$, $[X_s] \preceq [X_t]$ using the definition of Blackwell order. From the definition of $g_{\underline{u}}$, for all $t \in [0,1]$, we have $g_{\underline{u}}(X_s) = \underline{u}([X_1 \mid \mathcal{F}_s^X])$ almost surely where $\mathcal{F}^X$ denotes the filtration generated by X. By the definition of the concavification operator and since $\underline{u}$ is continuous, we have

$$\mathbb{E}[g_{\underline{u}}(X_s)] = \mathbb{E}[\underline{u}([X_1 \mid \mathcal{F}_s])] \leq \mathbb{E}[Cav(\underline{u})([X_1 \mid \mathcal{F}_s])] \leq Cav(\underline{u})([X_1]) \leq Cav(\underline{u})(\mu) \tag{1.7}$$

where the second inequality follows from Jensen's inequality (lemma 5.1) and the third from the fact that $Cav(\underline{u})$, being the pointwise limit of nondecreasing functions for the Blackwell order, is itself nondecreasing. Using Fubini's theorem, the right-hand side of (1.6) is less or equal than $Cav(\underline{u})(\mu)$. To prove the converse inequality, let $(\mu_i)_{i=1,..,e}$ in $\Delta(P)$ and $(\lambda_i)_{i=1,..,n}$ a convex combination such that

$$\sum_{i=1}^n \lambda_i\mu_i = \mu, \text{ and } \sum_{i=1}^n \lambda_i\underline{u}(\mu_i) \geq Cav(\underline{u})(\mu) - \varepsilon$$

Let $(S, Y_1, .., Y_n)$ be independent random variables such that $Y_i \sim \mu_i$ and $\mathbb{P}(S = i) = \lambda_i$. Define a martingale by the relation $X_1 = Y_S$ and $X_t = \sum_{i=1}^n \mathbb{E}[Y_i]\mathbb{1}_{S=i}$ for all $t \in [0,1)$. Then

$$\mathbb{E}[\int_0^1 g_{\underline{u}}(X_s)ds] = \int_0^1 \sum_{i=1}^n \lambda_i g_{\underline{u}}(\mathbb{E}[Y_i])ds = \int_0^1 \sum_{i=1}^n \lambda_i\underline{u}(\mu_i)ds \geq Cav(\underline{u})(\mu) - \varepsilon$$

and the proof follows by sending ε to zero [4]. For the second assertion, if the property is met, then X is clearly a maximizer. If X is a maximizer and this property is false, then we obtain a contradiction using the inequalities 1.7 and integrating with respect to s. □

Let us now prove the convergence of maximizers of the maximal variation problem to the maximizers of the continuous-time problem we just define. We need the following notation

NOTATION 2: *Given a discrete-time process $(L_1, .., L_n)$, the continuous-time version of this process is defined by*

$$\Pi_t^n = L_{\lfloor nt\rfloor} \quad \textit{for } t \in [0,1]$$

where $\lfloor a\rfloor$ is the greatest integer less or equal to a.

4. The ε is actually unnecessary but convenient for the presentation.

Proposition 1.8: *Let (L^n) be an asymptotically maximizing sequence of $\overline{\mathcal{V}}_n(\mu)$ in $\mathcal{M}_n(\mu)$, i.e. such that*

$$\frac{1}{n}\mathcal{V}_n(L^n, \mathcal{F}^{L^n}) \underset{n\to\infty}{\longrightarrow} Cav(\underline{u})(\mu).$$

Then the continuous-time versions of these martingales define a weakly relatively compact sequence of laws for the Meyer-Zheng topology (see [**45**]*) and any limit point belongs to*

$$\mathcal{P}_\infty(\mu) = \underset{[X]\in\mathcal{M}(\preceq\mu)}{argmax}\ \mathbb{E}[\int_0^1 g_{\underline{u}}(X_t)dt].$$

Proof. We refer to [**45**] for properties of the pseudo-paths or Meyer-Zheng topology on the Skorokhod space. This topology is defined on the set $\mathbb{D}([0,1],P)$ of càdlàg functions as the convergence in measure with respect to Lebesgue's measure (denoted λ) together convergence of the value at time 1 : a sequence y_n converges to y if

$$\forall \varepsilon > 0,\ \lambda(\{|y_n(x) - y(x)| \geq \varepsilon\}) \underset{n\to\infty}{\longrightarrow} 0, \qquad \text{and} \quad y_n(1) \underset{n\to\infty}{\longrightarrow} y(1)$$

The subset $\mathcal{M}(\preceq \mu)$ of laws of martingales (in $\Delta(\mathbb{D}([0,1],P))$ is weakly compact using theorem 2 in [**45**], the fact that the projection $(X_t)_{t\in[0,1]} \to X_1$ at time 1 is weakly-continuous and that condition $[X_1] \leq \mu$ is closed. Moreover, the functional $[X] \to \mathbb{E}[\int_0^1 g_{\underline{u}}(X_t)dt]$ is by construction weakly continuous. Let us denote Π^n denote the sequence of continuous-time versions of L^n, and notice that $[\Pi^n] \in \mathcal{M}(\preceq \mu)$ by construction. Assume, up to the extraction of some subsequence, that $[\Pi^n]$ converges weakly to some limit $[\Pi]$. Then, using the previous propositions,

$$\begin{aligned}
\frac{1}{n}\mathcal{V}_n((L_k)_{k=1,..,n}) &\leq \mathbb{E}[\sum_{k=1}^n \frac{1}{n}\underline{u}([L_k \mid L_1,..,L_{k-1}])] + \frac{1}{n}C_T\mathcal{V}_n^{N_1}((L_k)_{k=1,..,n}) \\
&= \mathbb{E}[\sum_{k=1}^n \frac{1}{n} g_{\underline{u}}(L_{k-1})] + \frac{1}{n}C_T\mathcal{V}_n^{N_1}((L_k)_{k=1,..,n}) \\
&= \mathbb{E}[\int_0^1 g_{\underline{u}}(\Pi_t^n)dt] + \frac{1}{n}C_T\mathcal{V}_n^{N_1}((L_k)_{k=1,..,n}) \\
&\underset{n\to\infty}{\longrightarrow} \mathbb{E}[\int_0^1 g_{\underline{u}}(\Pi_t)dt]
\end{aligned}$$

We conclude that $[\Pi] \in \mathcal{P}_\infty(\mu)$ since by assumption the left-hand side of the above inequality converges to the value of problem (1.6). $\square$

We point out in proposition 1.7 that the maximizers of the continuous-time problem are degenerate in the following sense. If we have $\underline{u}(\mu) < Cav(\underline{u})(\mu)$, then any optimal martingale jumps at the beginning to a position where $\mathbb{E}[g_{\underline{u}}(X_0)]) = Cav(\underline{u})(\mu)$ and then can either stop moving or evolve freely within a set on which $\underline{u}$ is linear. For the extremal situations, if $\underline{u}$ is strictly concave at μ, then the optimal martingale never moves and is uniquely determined. On the contrary, if $\underline{u}$ is linear on $\Delta(P)$, then any martingale will be optimal. It means that the asymptotic behavior of the optimal strategies of P1 is in general degenerate from the dynamic point of view.

2. Dual game

We introduce in this section an auxiliary game of complete information known in the finite case as the dual game, and that was at first introduced in De Meyer [**23**]. In order to simplify the proofs, we will consider in all this section an exchange game game (T, I, J) with a state variable in compact convex subset P of $\mathbb{R}^d$. This case contains the partially revealing game associated to a finite game introduced in the next section. We show that the results obtained in Laraki [**40**] can be generalized in our model, including the representation of the concavification of $\overline{u}$ as a the solution of a dual Hamilton-Jacobi equation. We deduce from this result a proof based on duality of the fact that the minmax of Γ_n divided by n converges to $Cav(\overline{u})$.

2.1. The model. We consider the game $\Gamma_n(\mu)$ defined by (T, I, J) where $\mu \in \Delta(P)$ for some compact convex subset P of $\mathbb{R}^d$. Given $\phi \in C(P, \mathbb{R})$, the dual game $\Gamma_n^*(\phi)$ is defined as follows. A strategy for P1 is a pair (μ, σ) where $\mu \in \Delta(P)$ and $\sigma \in \Sigma_n$. A strategy for P2 is some $\tau \in \mathcal{T}_n$. The payoff function is defined by

$$(\mu, \sigma, \tau) \to g_n^*(\phi, \mu, \sigma, \tau) = \mathbb{E}_{\Pi(\mu,\sigma,\tau)}[\langle L, \sum_{k=1}^{n} T(i_k, j_k)\rangle - \phi(L)]$$

Let us now define the maxmin and minmax of this game as

$$\underline{W}_n(\phi) = \underset{(\mu,\sigma)\in\Delta(P)\times\Sigma_n}{sup} \; \underset{\tau\in\mathcal{T}_n}{inf} \; g_n^*(\phi, \mu, \sigma, \tau) \tag{2.1}$$

$$\overline{W}_n(\phi) = \underset{\tau\in\mathcal{T}_n}{inf} \; \underset{(\mu,\sigma)\in\Delta(P)\times\Sigma_n}{sup} \; g_n^*(\phi, \mu, \sigma, \tau) \tag{2.2}$$

Our aim is to study the asymptotic behavior of the dual game, and precisely of $\overline{W}_n$.

NOTATION 3: *In the following, the convex conjugation denoted by $*$ applied to functions defined on $\Delta(P)$ or $C(P)$ refers to the duality between the space $M(P)$ of signed Radon measures on P and $C(P)$. All the functions defined on $\Delta(P)$ are extended on the whole space $M(P)$ by the value $+\infty$. Since $\Delta(P)$ is a weakly closed set in $M(P)$, if F is convex continuous on $\Delta(P)$, its extension is l.s.c. and convex and therefore $(F)^{**} = (F)$.*

The following result is a direct consequence of the definition.

PROPOSITION 2.1: *For all $\phi \in C(P)$ and $\mu \in \Delta(P)$*

$$\underline{W}_n(\phi) = (-\underline{V}_n)^*(-\phi) \qquad and \qquad \underline{V}_n(\mu) = -W_n^*(-\mu)$$

For all $\phi \in C(P)$ and $\mu \in \Delta(P)$

$$\overline{W}_n(\phi) \geq (-\underline{V}_n)^*(-\phi) \qquad and \qquad \overline{V}_n(\mu) \leq -W_n^*(-\mu)$$

NOTATION 4: *Any function $\phi \in C(P)$ is implicitly identified with its extension on $\mathbb{R}^d$ which is equal to ϕ on P and to $+\infty$ outside P. Its convex conjugate ϕ^* is therefore the usual convex conjugate of the extended function and if ϕ is convex on P, then $\phi = \phi^{**}$.*

LEMMA 2.1: *For all $\phi \in C(P)$*

$$\overline{W}_n(\phi) \leq \widehat{W}_n(\phi) \triangleq \underset{\tau\in\mathcal{T}_n^r}{inf} \; \underset{(i_1,..,i_n)\in I^n}{sup} \phi^*(\sum_{k=1}^{n} T(i_k, \tau_k))$$

where $T(i_k,\tau_k)=\int_J T(i_k,j_k)d\tau_k(i_1,..,i_{k-1})(j_k)$.

PROOF. At first we have from proposition 1.4

$$\overline{W}_n(\phi)\le \inf_{\tau\in\mathcal{T}_n^r}\ \sup_{(\mu,\sigma)\in\Delta(P)\times\Sigma_n} g_n^*(\phi,\mu,\sigma,\tau)=\inf_{\tau\in\mathcal{T}_n^r}\ \sup_{(\mu,\sigma)\in\Delta(P)\times\Sigma_n^r} g_n^*(\phi,\mu,\sigma,\tau)$$

since $\mathcal{T}_n^r\subset\mathcal{T}_n$ and P1 cannot obtain a better payoff against a reduced strategy of P2 by using non-reduced strategies. The set of strategies $(\mu,\sigma)\in\Delta(P)\times\Sigma_n^r$ can be identified to the set of joint distributions on $P\times I^n$. Moreover, the structure of reduced strategies allows to integrate at first the payoff function with respect to τ conditionally on $(i_1,..,i_n)$ and by Fubini theorem

$$\inf_{\tau\in\mathcal{T}_n^r}\ \sup_{(\mu,\sigma)\in\Delta(P)\times\Sigma_n^r} g_n^*(\phi,\mu,\sigma,\tau)=\inf_{\tau\in\mathcal{T}_n^r}\ \sup_{\pi\in\Delta(P\times I^n)}\mathbb{E}_\pi[\langle L,\sum_{k=1}^n T(i_k,\tau_k)\rangle-\phi(L)]$$

The supremum over $\Delta(P\times I^n)$ is then equal to the supremum over $P\times I^n$. The supremum over P being by definition $\phi^*(\sum_{k=1}^n T(i_k,\tau_k))$, this concludes the proof. □

A recurrence formula.

Let $Cv(\mathbb{R}^d)$ denote the set of continuous convex functions f on $\mathbb{R}^d$.

LEMMA 2.2: *The operator H below defines a map from $Cv(\mathbb{R}^d)$ to itself.*

$$H(f)(x)=\inf_{\tau\in\Delta(J)}\ \sup_{i\in I} f(x+T(i,\tau))$$

and for all $\varepsilon>0$, there exists Borel measurable function

$$\tau_\varepsilon:\mathbb{R}^d\to\Delta(J)$$

which is ε-optimal in the sense

$$\forall x\in\mathbb{R}^d,\ \sup_{i\in I} f(x+T(i,\tau_\varepsilon(x)))\le H(f)(x)+\varepsilon$$

PROOF. $H(f)$ is real-valued since T is bounded and f continuous. For the convexity of $H(f)$ let $x_1,x_2\in\mathbb{R}^d$ and $\lambda\in[0,1]$. Let τ_1,τ_2 be η-optimal for the problems $H(f)(x_1)$ and $H(f)(x_2)$, then $\lambda\tau_1+(1-\lambda)\tau_2$ is η-optimal for the problem $H(f)(\lambda x_1+(1-\lambda)x_2)$ using the linearity of the integral and the convexity of f. The conclusion follows by sending η to zero. For the functions τ_ε, we proceed by discretization. For any fixed τ, the function

$$x\to\sup_{i\in I} f(x+T(i,\tau))$$

is continuous since it is locally bounded and convex. Since so is $H(f)$, any $\varepsilon/2$ optimal τ in x is therefore ε-optimal in a neighborhood of x. It follows that there exists a countable measurable partition of $\mathbb{R}^d$ and a function τ_ε constant on the element of the partition having the required properties. □

We prove in the following proposition that $\widehat{W}_n$ satisfies a recurrence formula.

PROPOSITION 2.2: *For all $\phi\in C(P)$*

$$\widehat{W}_n(\phi)=H^n(\phi^*)(0),\quad \textit{with}\quad H^n=\underbrace{H\circ\ldots\circ H}_{n}$$

PROOF. By induction, it is sufficient to prove that

$$\widehat{W}_n(\phi) = \inf_{\tau \in \mathcal{T}^r_{n-1}} \sup_{i_1,..,i_{n-1}} H(\phi^*)(x)$$

with

$$x = \sum_{k=1}^{n-1} T(i_k, \tau_k)$$

For any reduced strategy $\tau \in \mathcal{T}^r_n$, we have the following inequality

$$\begin{aligned} \sup_{i_1,..,i_n} \phi^*(\sum_{k=1}^{n} T(i_k, \tau_k)) &= \sup_{i_1,..,i_n} \phi^*(x + T(i_n, \tau_n(i_1, .., i_{n-1}))) \\ &\geq \sup_{i_1,..,i_{n-1}} H(\phi^*)(x) \end{aligned}$$

which allows to prove a first inequality by taking the infimum over τ on both sides. For the reverse inequality, take $\tau_n = \tau_\varepsilon(x)$ with τ_ε given by the previous lemma for $f = \phi^*$. For any $(\tau_1, .., \tau_{n-1}) \in \mathcal{T}^r_{n-1}$, the above definition of τ_n defines a reduced strategy in $\mathcal{T}^r_n$ and we have

$$\phi^*(\sum_{k=1}^{n} T(i_k, \tau_k)) = \phi^*(x + T(i_n, \tau_\varepsilon(x))) \leq H(\phi^*)(x) + \varepsilon$$

which allows easily to conclude by taking successively the supremum over $(i_1, .., i_{n-1})$ and the infimum over $(\tau_1, ..\tau_{n-1})$ and then by sending ε to zero. □

The dual PDE formulation. Let us define $\psi_n(\mu) = \frac{1}{n}\overline{V}_n(\mu)$. As for the previous dual inequality, we have

$$\begin{aligned} (-\psi_n)^*(-\phi) &= \sup_{\mu \in \Delta(P)} \frac{1}{n}\overline{V}_n(\mu) - \langle \phi, \mu \rangle = \frac{1}{n} \sup_{\mu \in \Delta(P)} \inf_{\tau \in \mathcal{T}_n} \sup_{\sigma \in \Sigma_n} g^*_n(n\phi, \mu, \sigma, \tau) \\ &\leq \frac{1}{n}\overline{W}_n(n\phi) \leq \frac{1}{n}\widehat{W}_n(n\phi) = \frac{1}{n} H_n((n\phi)^*)(0) \end{aligned}$$

The above expression suggests to introduce the following family of operators on the set $C(\mathbb{R}^d)$ of continuous functions on $\mathbb{R}^d$.

$$R_\delta(f)(x) = \inf_{\tau \in \Delta(J)} \sup_{i \in I} f(x + \delta T(i, \tau))$$

It is easily seen that $\frac{1}{n}H^n((n\phi)^*)(0) = R^n_{1/n}(\phi^*)$.

Let also $BUC(\mathbb{R}^d)$ denote the set of bounded uniformly continuous functions on $\mathbb{R}^d$ endowed with the uniform norm and for Lipschitz functions, let $Lip(f)$ denote the Lipschitz constant of f.

LEMMA 2.3: *The family of operators $(R_\delta)_{\delta \geq 0}$ maps $BUC(\mathbb{R}^d)$ into itself and has the following properties*

1) For all $f \in BUC(\mathbb{R}^d)$, $R_0(f) = f$

2) For all $f \in BUC(\mathbb{R}^d)$, the function $\delta \to R_\delta(f)$ is continuous .

3) There exists a constant C_1 such that $\|R_\delta(f)\|_\infty \leq C_1\delta + \|f\|_\infty$

4) For all $f, g \in BUC(\mathbb{R}^d)$, $\alpha \in \mathbb{R}$, $R_n(f + \alpha) = R_n(f) + \alpha$

5) For all $f, g \in BUC(\mathbb{R}^d)$, $\|R_\delta(f) - R_\delta(g)\|_\infty \leq \|f - g\|_\infty$

6) For all $f \in BUC(\mathbb{R}^d)$ which is Lipschitz, $Lip(R_\delta(f)) \leq Lip(f)$ and there exists a constant C_2 depending only on $Lip(f)$ such that $\|R_\delta(f) - f\| \leq C_2\delta$.

7) There exists a constant C_3 such that for all $f \in BUC(\mathbb{R}^d)$ of class C^2 with bounded derivatives, we have

$$\left|\frac{R_\delta(f) - f}{\delta} - g_{\overline{u}}(\nabla f)\right| \leq C_3\delta(\|\nabla^2 f\|_\infty)$$

where $g_{\overline{u}}(x) = \overline{u}(\delta_x)$

Proof. The first six points are obvious. For the last one, we have using the second order Taylor expansion of f

$$f(x + \delta T(i,\tau)) = f(x) + \delta\langle\nabla f(x), T(i,\tau)\rangle + \delta^2 R(x,i,\tau)$$

where $|R(x,i,\tau)| \leq C\|\nabla^2 f\|_\infty$ for some constant C depending only on T. It follows that

$$\left|\underset{\tau\in\Delta(J)}{inf}\ \underset{i\in I}{sup}\ f(x+\delta T(i,\tau)) - \underset{\tau\in\Delta(J)}{inf}\ \underset{i\in I}{sup}\left[f(x) + \delta\int_J \langle\nabla f(x), T(i,j)\rangle d\tau(j)\right]\right|$$

$$\leq C\delta^2\|\nabla^2 f\|_\infty$$

which concludes the proof since from the definition of $\overline{u}$

$$\underset{\tau\in\Delta(J)}{inf}\ \underset{i\in I}{sup}\int_J \langle\nabla f(x), T(i,j)\rangle d\tau(j) = g_{\overline{u}}(\nabla f(x))$$

□

Lemma 2.4: *The function $g_{\overline{u}}$ is positively homogenous and Lipschitz on $\mathbb{R}^d$.*

Proof. This follows directly from the properties of $\overline{u}$. □

These properties allow us to apply the results on approximation schemes of Souganidis [**55**] as in Laraki [**40**].

Theorem 2.1: *For all convex Lipschitz functions f on $\mathbb{R}^d$, $R^n_{1/n}(f)(0) \to W(f)$ where $W(f) = \chi(1,0)$ with χ the unique viscosity solution of the following Hamilton-Jacobi equation*

$$\begin{cases} \frac{\partial\chi}{\partial t}(t,x) - g_{\overline{u}}(\nabla_x\chi(t,x)) = 0 & \text{for } (t,x) \in (0,1] \times \mathbb{R}^d \\ \chi(0,x) = f(x) & \text{for } x \in \mathbb{R}^d \end{cases} \tag{2.3}$$

in the class of uniformly continuous functions. Moreover, there exists a constant C_0 depending uniquely on the Lipschitz constant of f such that

$$|R_n(f)(0) - \chi(1,0)| \leq \frac{C_0}{\sqrt{n}}$$

Proof. That the solutions are unique within the class of uniformly continuous functions for the considered equation follows from Bardi and Evans [**5**] (theorem 3.1), we always consider these solutions in the following. Note that if χ is the solution to (2.3) and $\alpha \in \mathbb{R}$, then $\chi + \alpha$ is the solution to the same equation with boundary condition $\chi(0,.) = f + \alpha$. Using property 4 in lemma 2.3, this allows to assume that $f(0) = 0$. For all n, $R^n_{1/n}(f)(0)$ depends only of the restriction of f to the ball $B(0, C_T)$. Therefore f can be replaced by a truncation $\beta_b(f)$ with β_b defined by

$$\beta_b : \mathbb{R} \to \mathbb{R} : x \to \begin{cases} b & \text{if} \quad x \geq b \\ x & \text{if} \quad |x| \leq b \\ -b & \text{if} \quad x \leq -b \end{cases}$$

for some sufficiently large b. Lemma 2.3 and 2.4 allows us to apply the introductory theorem in [**55**] which implies that $R^n_{1/n}(f)(0) \to \chi_b(1,0)$ where χ_b is the unique viscosity solution of the following Hamilton-Jacobi equation

$$\begin{cases} \frac{\partial \chi}{\partial t}(t,x) - g_{\overline{u}}(\nabla_x \chi(t,x)) = 0 & \text{for } (t,x) \in (0,1] \times \mathbb{R}^d \\ \chi(0,x) = \beta_b(f(x)) & \text{for } x \in \mathbb{R}^d \end{cases} \tag{2.4}$$

Moreover, the cited theorem asserts that there exists a constant C_0 which depends only on $\|\beta_b(f)\|_\infty$ and $Lip(\beta_b(f))$ such that

$$|R^n_{1/n}(f)(0) - \chi_b(1,0)| \leq \frac{C_0}{\sqrt{n}}$$

for n large enough (the bound depending also only on the same constants (see [**55**] p.21). Since these two quantities are bounded by $(C_T \vee 1)Lip(f)$, this constant C_0 depends only on $Lip(f)$ and is independent of b. Using that $g_{\overline{u}}$ is positively homogenous and Lipschitz, proposition 3.1 in [**7**] shows that $\chi_b = \beta_b(\chi)$ where χ is the unique solution of (2.3) (with boundary condition f). We can therefore fix b sufficiently large so that $\chi(1,0) = \chi_b(1,0)$ which concludes the proof. □

Gathering the preceding results, we obtain

Corollary 1: *There exists a constant C_0 such that for all $\phi \in C(P)$,*

$$\frac{1}{n}\overline{W}_n(\phi) \leq W(\phi^*) + \frac{C_0}{\sqrt{n}}$$

Proof. Apply the preceding result to all the function ϕ^* for $\phi \in C(P)$. Since for $\phi \in C(P)$, ϕ^* is C_P-Lipschitz where $C_P = \sup_{x \in P} |x|$ the constant C_0 does not depend on ϕ. □

Let us now relate this result with the study of the primal game Γ_n. For this, let us recall Hopf's formula, which gives an explicit expression for the solution of (2.3).

Proposition 2.3: *For all Lipschitz functions $f \in Cv(\mathbb{R}^d)$, $W(f) = (f^* - g_{\overline{u}})^*(0)$.*

Proof. The solution χ is given by the Hopf's formula ([**5**] theorem 3.1)

$$\chi(t,x) = \sup_{p \in \mathbb{R}^d} \inf_{q \in \mathbb{R}^d} [f(q) + \langle p, x - q \rangle + t g_{\overline{u}}(p)].$$

And a direct computation shows that $\chi(1,0) = (f^* - g_{\overline{u}})^*(0)$. □

This Hopf's function is actually related to the concave conjugate of $\overline{u}$.

Lemma 2.5: *The conjugate of the function $\overline{u}$ defined on the set $C(P)$ verifies*

$$\forall \phi \in C(P), \quad (-\overline{u})^*(-\phi) = W(\phi^*)$$

Proof. It follows from the definition that

$$\begin{aligned}(-\overline{u})^*(-\phi) &= \underset{\mu\in\Delta(P)}{sup}\ \overline{u}(\mu) - \langle \phi, \mu\rangle \\ &= \underset{x\in P}{sup}\left(\underset{\mu\,:\,\mathbb{E}(\mu)=x}{sup}\ g_{\overline{u}}(x) - \langle \phi, \mu\rangle\right) \\ &= \underset{x\in P}{sup}\ g_{\overline{u}}(x) - \phi^{**}(x) = (\phi^{**} - g_{\overline{u}})^*(0) = W(\phi^*)\end{aligned}$$

where the third equality follows from the continuity of ϕ which implies

$$\underset{\mu\,:\,\mathbb{E}(\mu)=x}{inf}\ \langle \phi, \mu\rangle = \phi^{**}(0)$$

□

We are now able to prove the dual version of the "Cav(u)" theorem.

Proposition 2.4: *For all $\mu \in \Delta(P)$*

$$Cav(\overline{u})(\mu) \leq \frac{1}{n}\overline{V}_n \leq Cav(\overline{u})(\mu) + C_0 n^{-1/2}$$

Proof. The first inequality is proved as in proposition 1.5. For the second one, let us define $\psi_n(\mu) = \frac{1}{n}\overline{V}_n(\mu)$. We have

$$\begin{aligned}(-\psi_n)^*(-\phi) &= \underset{\mu\in\Delta(P)}{sup}\frac{1}{n}\overline{V}_n(\mu) - \langle \phi, \mu\rangle = \frac{1}{n}\underset{\mu\in\Delta(P)}{sup}\ \underset{\tau\in\mathcal{T}_n}{inf}\ \underset{\sigma\in\Sigma_n}{sup}\ g_n^*(n\phi, \mu, \sigma, \tau) \\ &\leq \frac{1}{n}\overline{W}_n(n\phi) \leq R^n_{\frac{1}{n}}(\phi^*)(0) \leq W(\phi^*) + C_0 n^{-1/2}\end{aligned}$$

Using that $(-\overline{u})^*(-\phi) = W(\phi^*)$, the preceding relation implies

$$\psi_n(\mu) \leq -(-\overline{u})^{**}(\mu) + C_0 n^{-1/2}.$$

Using that $\overline{u}$ is concave and weakly continuous, we have that $-(-\overline{u})^{**}(\mu) = Cav(\overline{u})$ and this concludes the proof. □

Let us also mention the following corollary

Corollary 2: *If the non-revealing game has a value for all $x \in P$ (or equivalently for all $\mu \in \Delta(P)$), then*

$$Cav(u)(\mu) \leq \frac{1}{n}\underline{V}_n(\mu) \leq \frac{1}{n}\overline{V}_n(\mu) \leq Cav(u)(\mu) + C_0 n^{-1/2}$$

where $u = \underline{u} = \overline{u}$.

3. Finite games.

3.1. A first identification. In the following, for a finite set K, its cardinal is also denoted K and $\Delta(K)$ denotes the set of probabilities over K. A finite Game $G_1(p)$ with incomplete information on one side à la Aumann-Maschler is represented by (I, J, K, A). I, J, K are finite sets which are respectively the action sets of P1,P2 and the state space. The last term A

represents the payoff function and is a matrix of real numbers $(A_{i,j}^k)_{(i,j,k)\in I\times J\times K}$. p is an element of $\Delta(K)$. Formally, the value of such a game is given by

$$v_1 : \Delta(K) \to \mathbb{R} : p \to \underset{\sigma\in\Delta(I)^K}{sup} \ \underset{\tau\Delta(J)}{inf} \sum_{i,j,k} p^k \sigma^k(i)\tau(j)A_{i,j}^k$$

As usual, the set $\Delta(K)$ is identified in this formula to the $(K-1)$-dimensional simplex in $\mathbb{R}^K$ which will be denoted S_K when needed in order to avoid confusions. We will now introduce another identification which is different from this classical one. Let $(e_k, k = 1,..,K)$ denote the canonical basis of $\mathbb{R}^K$. At first, we can identify K with the subset $\{e_k, k = 1,..,K\}$ of $\mathbb{R}^K$. This map induces naturally a linear isomorphism h between $\Delta(K)$ and the set $\Delta(\{e_k, k = 1,..,K\}) \subset \Delta(\mathbb{R}^K)$ of probabilities over $\mathbb{R}^K$ supported by $\{e_k, k = 1,..,K\}$. Precisely,

$$h : p = (p^k)_{k=1,..,K} \in \Delta(K) \ \longrightarrow \sum_{k=1}^K p_k \delta_{e_k}$$

where δ_{e_k} is the Dirac mass in e_k. Using this identification, the sum over k appearing in the definition of the value v_1 can be seen as the expectation with respect to some $\mathbb{R}^K$-valued random variable L of law $h(p)$. σ can be extended to any function $\tilde{\sigma}$ defined on the whole space $\mathbb{R}^K$ such that $\tilde{\sigma}(e_k) = \sigma^k$. Denoting $T(i,j) = (A_{i,j}^k)_{k=1,..,K}$, we obtain

$$\sum_{i,j,k} p^k \sigma^k(i)\tau(j)A_{i,j}^k = \mathbb{E}_{\Pi(h(p),\tilde{\sigma},\tau)}[\langle L, T(i,j)\rangle] \tag{3.1}$$

(T, I, J) defines a game $\widehat{G}_1(\mu)$ in the class of linear games introduced above. Let also G_n and $\widehat{G}_n$ denote the n-times repeated games in behavioral strategies associated to G_1 and $\widehat{G}_1$. The next results shows that our extension has the required properties.

Proposition 3.1: *For all $\mu \in \Delta_1(\mathbb{R}^K)$, the value $V_n(\mu) = \overline{V}_n(\mu) = \underline{V}_n(\mu)$ of the game $\widehat{G}_n(\mu)$ exists and*

$$\forall p \in \Delta(K), \quad V_n(h(p)) = v_n(p).$$

Proof. Since the expectation (3.1) does not depend on the choice of the function $\tilde{\sigma}$, the value V_1 of $\widehat{G}_1$ exists on $h(\Delta(K))$ and is such that for all $p \in \Delta(K)$, $V(h(p)) = v(p)$. The same argument also works for G_n. It remains to prove that the value of $\widehat{G}_n(\mu)$ exists for all μ. We will show that this game has a value in reduced strategies which is sufficient using proposition 1.4. The set of reduced strategies of P1 is the set of joint laws π over $\mathbb{R}^K \times I^n$ having for marginal μ on $\mathbb{R}^K$. The set of reduced strategies of P2 is the set of sequences $\tau = (\tau_1,..,\tau_n)$ with $\tau_q \in \Delta(J)^{(q-1)I}$ for $q = 1,..,n$. These two sets are weakly compact and the payoff function is bilinear and weakly continuous with respect to (π,τ) so that the result follows from the minmax theorem of Sion [**53**]. □

Note that actually all the preceding results apply for the extended finite game $\widehat{G}_n(\mu)$ on $\Delta(S_K)$. Moreover, the dual inequalities mentioned in the previous section are equalities. Precisely, the following is a direct consequence of the minmax theorem.

Proposition 3.2: *Let $\widehat{G}_n$ denote the partially revealing game extending G_n and $\widehat{G}_n^*(\phi)$ the associated dual game defined for $\phi \in C(S_K)$. Then, $\widehat{G}_n^*(\phi)$ has a value $W_n(\phi)$ and*

$$(-V_n)^*(-\phi) = W_n(\phi)$$

Moreover $\frac{1}{n}W_n(\phi) \to W(\phi^*)$.

3.2. The partially revealing game. The above identification is not introduced here as a remark but was already noticed to be useful for the study of repeated games in the work of De Meyer an Marino [**29**] where an extended game called the partially revealed game was introduced. The authors defined this game as the modification of the game $G_n(p)$ where P1, instead of being informed of the true state that has been selected in K, only receives a random signal s. Since only the conditional law of k given s is relevant, we can assume that the state space is $\Delta(K)$. Indeed, in this game the joint law of (k, s) is known and this amounts to assume that Nature selects at random a variable in $\Delta(K)$ using some probability distribution over $\Delta(K)$. This extended game is therefore defined on $\Delta(\Delta(K)) = \Delta(S_K)$ which is identified as the subset of $\Delta(\mathbb{R}^K)$ of probabilities supported by the $(K-1)$-dimensional simplex S_K. It is easily seen that the formal definitions of $\widehat{G}$ and of the partially repeated game coincide on $\Delta(\Delta(K))$. To see this, just note that for a fully informative signal s, the conditional law of k given s is the Dirac mass δ_k. The game $G_1(p)$ is therefore identified to $\widehat{G}_1(h(p))$ as in the previous section. Despite the fact that the partially repeated game has a more intuitive meaning, we chose to keep the extension $\widehat{G}$ as a definition in order to unify our results with the model of financial exchange games detailed in the next section.

3.3. Discussion and an open question. The structure of the partially revealing game allows to state a representation formula as the value function of a maximal variation problem, which holds even for $\mu = h(p)$. Using the former notations, we have for the value of $G_n(p)$

$$v_n(p) = \underset{[(L_k)_{k=1,..,n}] \in \mathcal{M}_n(h(p))}{sup} \mathbb{E}[\sum_{k=1}^{n} V_1([L_k \mid \mathcal{F}^L_{k-1}])]$$

where v_n is the value of the initial finite game G_n and V_1 the value of $\widehat{G}_1$. In order to avoid confusions, let us denote $w(p)$ the value of the non-revealing game associated to $G_1(p)$ and u the value of the non revealing game associated to $\widehat{G}_1(\mu)$ as in the previous sections. With these notations, for all $p \in \Delta(K)$, $w(p) = u(h(p))$. Moreover, $Cav_{\Delta(K)}(w)(p)$ is equal to $Cav_{\Delta(S_K)}(u)(h(p))$ since $h(\Delta(K))$ is a face of $\Delta(S_K)$. The results of Cardaliaguet in [**17**] imply that that the function $Cav(w)(p)$ is the unique viscosity solution of the following first-order Hamilton-Jacobi equation with obstacle in $\Delta(K)$

$$max\{w(p) - \phi\, ;\, \lambda_{max}(\nabla^2 \phi)\} = 0 \tag{3.2}$$

There is therefore a PDE formulation of $Cav(u)$ on a finite-dimensional simplex of $\Delta(S_K)$. It is shown in [**16**] that being solution of the above equation is equivalent to being a dual solution of some auxiliary equation. In our (very) particular case, we have the following dual representation. The “conjugate” of $Cav(w)$ in the sense $y \to (-Cav(w)^*(-y))$ is the value in $(t, x) = (1, 0)$ of the solution of (2.3) with boundary condition

$$f_y(x) = \underset{k \in K}{sup}\,(x - y)^k.$$

(see proposition 3.1 in [**17**] together a change of variable, and also [**40**]). The results of the previous section on the dual game applied to $\widehat{G}^*_n$ show that this dual representation is still true on the whole set $C(S_K)$. To see this, note that the relation

$$(-Cav(u))^*(-\phi) = (-u)^*(-\phi) = W(\phi^*) \tag{3.3}$$

relates the conjugate of $Cav(u)$ to the value in $(1, 0)$ of the solution of (2.3) with boundary condition ϕ^*. That this result is really an extension follows from the fact that f_y is the convex conjugate of the linear function $x \to \langle y, x\rangle$ (restricted to S_K). The dual representation given in Laraki [**40**] appears therefore as the equation (3.3) restricted to the finite-dimensional subspace of linear functions in $C(S_K)$.

Finally, let us mention an open question. Despite the dual formulation and the martingale representation (proposition 1.7 have been extended to the whole sets $C(S_K)$ and $\Delta(S_K)$, the question of the formulation of a primal PDE generalizing (3.2) on $\Delta(S_K)$ is still opened. However, due to the relationship outlined in [**18**] between the primal and dual formulations of some PDE problems arising in differential games and recent works (for example Cardaliaguet and Souquiere [**19**]) dealing with PDE in measure spaces, we think (and hope) that this formulation will appear in some future works. Let us also mention that exactly the same situation will be observed for the second order analysis exposed in the next chapter.

4. The financial exchange models

Let us now present briefly the model that was the first motivation for studying the problems of maximal variation. It will be studied in greater details in chapter 3.

4.1. A financial exchange game. Consider two players (P1 and P2) exchanging a risky asset A against a numéraire asset N. The liquidation value at date $t = 1$ of the asset A is denoted L and the liquidation value of N is assumed to be equal to 1. Exchange occurs repeatedly during n rounds from date $t = 0$ up to date $t = 1$. At the beginning of the game, the price L is drawn according to some probability μ over $\mathbb{R}$. P1 is informed of the realization of L while P2 knows only μ and that his opponent is informed. Each transaction round is a zero-sum exchange game described by an abstract exchange mechanism $T = (R, N)$. The two players have Polish actions sets I,J and T is a Borel mapping from $I \times J$ to $\mathbb{R}^2$ that represents what P1 receives from P2. Precisely, if the players choose the pair of actions (i, j), then P1 will receive from P2 $R(i, j)$ shares of asset A and the quantity $N(i, j)$ in Numéraire (typically one is positive and the other negative). If portfolio of P1 after round k is denoted $y_k = (y_k^A, y_k^N)$ and (i_k, j_k) the pair of actions played at round k, then

$$y_k = y_{k-1} + T(i_k, j_k)$$

This repeated game in behavioral strategies is denoted $E_n(\mu)$. Strategies are defined exactly as in section 1. Players are assumed to be risk-neutral and to have sufficiently large initial endowments so that the constraints $y_k \geq 0$ and $z_k \geq 0$ are ignored. This amounts to assume that the initial portfolios are such that $y_0 = 0$, $z_0 = 0$. $E_n(\mu)$ is then a zero-sum game, and P1's payoff is the expectation of his final portfolio given by

$$\mathbb{E}_{\Pi(\mu,\sigma,\tau)}[\langle y_n^L, L\rangle + y_n^N]$$

Note that this payoff is an affine function of the state variable L.

The Price Process. Since the liquidation value at date 1 of the risky asset A is L, the valuation of this asset by a risk-neutral agent round k is simply the expected value of L given the available information. Let us define the price after round k as the valuation of the risky asset for the uniformed player P2. Available information after round k is given by the σ-field $\mathcal{F}_k = \sigma(i_1, j_1, ..., i_k, j_k)$ of past observations. Formally, our price process is the martingale $L_k = \mathbb{E}_\Pi[L \mid \mathcal{F}_k]$.

4.2. Identification to a linear game. Defining $\widetilde{L} = (L, 1) \in \mathbb{R}^{d+1}$, the above payoff function is linear with respect to the state variable $\widetilde{L}$, equal to $\mathbb{E}[\langle \widetilde{L}, \sum_{k=1}^n y_k \rangle]$. This is the payoff function of the linear game $\Gamma_n(\mu \otimes \delta_1)$ defined by T, I, J. Using this identification, the game we just described is a linear game as defined in section 1 with state variable $\widetilde{L}$. The set of admissible distributions for $\widetilde{L}$ is just restricted to distributions of the type $\widetilde{\mu} \triangleq \mu \otimes \delta_1$.

The price process $(L_1, .., L_n)$ in E_n we just defined is therefore the first coordinate of the expected belief martingale of P2 in the extended linear game. Since this linear game is restricted to distributions of the type $\mu \otimes \delta_1$, the second coordinate of any martingale of expected beliefs in the extended game $\Gamma_n(\widetilde{\mu})$ is always constant and equal to 1. It is not difficult in this context to traduce the results given in proposition 1.2 and 1.3 in terms of the truncated martingale $(L_1, .., L_n)$.

4.3. Invariance hypotheses. Let us now present the set of hypotheses given in [**25**] on the financial game model to obtain asymptotic results on the equilibrium price processes.

The main result in [**25**] concerns the asymptotic of the price process at equilibrium. Suppose that the above described game has a value $v_n(\mu)$ for all $\mu \in \Delta^2(\mathbb{R})$ and that both players have optimal strategies. Let us consider a sequence of price processes induced by a some pair of optimal strategies in the game $E_n(\mu)$ denoted $(L^n_k)_{k=0,...,n}$. The n transaction rounds occur between the date $t = 0$ when P1 receives the message the date $t = 1$. Assuming that round k occurs at time $t = \frac{k}{n}$ we can then extend this price process in a continuous time process Π^n_t, piecewise constant on the intervals $[\frac{k}{n}, \frac{k+1}{n})$. Precisely, with $\lfloor a \rfloor$ the greater integer less or equal to a, and $L^n_0 = \mathbb{E}(L)$, we define:

$$\forall t \in [0, 1] \quad \Pi^n_t = L^n_{\lfloor nt \rfloor}$$

When n becomes large, the time between two transaction rounds tends to zero, and the price process appears naturally as an approximation of a continuous-time price process. The limit, if it exists, of the processes Π^n_t, represents then a continuous-time "equilibrium" price process. From proposition 1.2, the problem of identifying the set of limit equilibrium price processes is equivalent to the study of the asymptotic behavior of sequences of martingales of length n that maximize the functional $\mathcal{V}^{v_1}_n$ in $\mathcal{M}_n(\mu)$. This type of problems will be intensively explored in the next chapter.

The main result in [**25**] is that, under some hypotheses on the exchange mechanism, Π^n converges in distribution to some continuous-time martingale depending only on the law μ, called continuous martingale of maximal variation (CMMV) (see chapter 3 for a precise statement).

The hypotheses on the exchange mechanism are the following

H1) *Existence of a value*: For all n and μ the game $E_n(\mu)$ has a value $v_n(\mu)$ and both players have optimal strategies.

H2)* *Continuity* : The value v_1 is Lipschitz with respect to the Wasserstein distance of order p for a $p \in [1, 2)$. This property is always true in the models we consider with $p = 1$, but it was introduced for games in which T is not bounded (see [**25**] and also chapter 3 for an example of game with possibly unbounded payoff).

H3) *Invariance with respect to the numéraire scale*

$$\forall \alpha \geq 0,\ \forall X \sim \mu \in \Delta^2(\mathbb{R}), \quad v_1([\alpha X]) = \alpha v_1([X]).$$

It means that replacing for example euros by dollars does not affect the value.

H4) *Invariance with respect to the risk-less part of the risky asset*

$$\forall \beta \in \mathbb{R}, \forall X \sim \mu \in \Delta^2(\mathbb{R}), \quad v_1([X + \beta]) = v_1([X]).$$

It means that replacing L by $L + x$ where x is some constant (a non-risky asset), does not affect the value of the game

H5) *Positive value of information*: v_1 is nonnegative and there exists μ such that $v_1(\mu) > 0$.

Remark 4.1: *Let us denote V_1 the value of the extended game Γ_1 with state variable $\widetilde{L}$ which exists using $H1$ for all distributions of type $\mu \otimes \delta_1$ (and their dilatations) since it is equivalent to $E_1(\mu)$. We have moreover the straightforward relation*

$$\forall \mu \in \Delta^2(\mathbb{R}), \quad V_1(\mu \otimes \delta_1) = v_1(\mu)$$

*The hypothesis $H3$ does **not** follow from the positive homogeneity of V_1 (lemma 1.1). Indeed, let $(X, 1)$ be a random variable of law $\mu \otimes \delta_1$, then we only have that*

$$V_1([\alpha X, \alpha 1]) = \alpha V_1([X, 1]) = \alpha v_1([X])$$

$$\textit{while} \quad v_1([\alpha X]) = V_1([\alpha X, 1]).$$

Note that the value of the non-revealing game constructed from Γ_1 has a value denoted u for all the distributions, which equals the value of the non-revealing game denoted w constructed from E_1 and that

$$\forall \mu \in \Delta^2(\mathbb{R}), \quad u(\widetilde{\mu}) = w(\mu) = v_1(\delta_{\mathbb{E}(\mu)})$$

4.4. Second order analysis. From the results of section 1, the asymptotic behavior of V_n is described by the $Cav(u)$-theorem, and the first term in the asymptotic expansion of V_n is $nCav(u)$. As shown for the case of the partially revealing game, if u is not linear, then the asymptotic behavior of the optimal expected belief martingales in the problem of maximal variation is degenerate. The results developed in [**25**] concern the second term in this asymptotic expansion which is of order $\sqrt{n}$ and appears only, in the context of finite games, when u is linear. The hypothesis that brings us to this case in this financial model is $H4$. This hypothesis means that adding a non-risky asset to the original one does not affect the value of the game, implying that P1 can take benefit only of the asymmetry of information.

By linearity, for a law μ, $w(\mu) = v_1(\delta_x)$ where δ_x is the Dirac mass at the point $x = \mathbb{E}_\mu[X]$. Moreover, the continuity of v_1 $(H2)$ implies that, with $X \sim \mu$ and if $x = 0$

$$v_1(\delta_0) = \underset{\lambda\to 0,\, \lambda>0}{lim} v_1([\lambda X]) = \underset{\lambda\to 0,\, \lambda>0}{lim} \lambda v_1([X]) = 0$$

Therefore, using this invariance property $v_1(\delta_x) = 0$ for all x and this in turn implies $w = 0$ and $Cav(w) = 0$ (and $Cav(u) = 0$ on the face of admissible laws $\widetilde{\mu}$). In such a game, contrary to the case of non-linear functions u, the optimal behavior of the informed player will be to reveal progressively his information.

4.5. The translation invariance hypothesis. The invariance by translation hypothesis $H4$ allows us to reformulate the problem of maximal variation given in proposition 1.2 in order to obtain the same type of functional as the ones introduced in [**43**] and [**25**]. Precisely, for a linear game Γ_n defined by I, J, T, assume

$$(H'4)\ \forall \beta \in \mathbb{R}^d, \forall X \sim \mu \in \Delta^1(\mathbb{R}^d), \quad \underline{V}_1([X+\beta]) = \underline{V}_1([X])$$

then for all $\mu \in \Delta^1(\mathbb{R}^d)$ and all martingales $(L_k, \mathcal{F}_k)_{k=1,..,n} \in \mathfrak{M}_n(\mu)$

$$\mathcal{V}_n((L_k, \mathcal{F}_k)_{k=1,..,n}) = \mathbb{E}[\sum_{k=1}^{n} \underline{V}_1([L_k - L_{k-1} \mid \mathcal{F}_{k-1}])] \tag{4.1}$$

since $H'4$ implies $\underline{V}_1([L_k \mid \mathcal{F}_{k-1}]) = \underline{V}_1([L_k - L_{k-1} \mid \mathcal{F}_{k-1}])$. In order to study the optimization problem defined in proposition 1.2, it is then equivalent to know the function $\underline{V}_1$ on the subset of centered probabilities, and to take (4.1) as definition for the $\underline{V}_1$-variation.

The next chapter is devoted to a general study of this kind of problems of maximal variation over martingales, which are from the previous discussion intimately related to the asymptotic study of linear games under the hypothesis $H'4$.

5. Technical proofs.

Proof of lemma 1.2. At first, this statement is equivalent to the existence of a measurable function

$$H_\varepsilon : \mathbb{R}^d \times \Delta^1(\mathbb{R}^d) \to \Delta(I)$$

having the property that the strategy $L \to H_\varepsilon(L, \mu)$ is ε-optimal in $\Gamma_1(\mu)$ using the correspondence $\phi_\varepsilon(L, \mu, U) = \Phi(H_\varepsilon(L, \mu), U)$ where Φ is given by theorem 1.3 in the appendix. The main problem here is that there is no easily tractable topology on the set of P1 strategies (usual Young topologies require a reference measure μ which will be a variable for us), but thanks to the regularity of the payoff function with respect to the state variable, we can proceed by a standard (but quite long) discretization method. Let us fix some $0 < \eta < 1$. Let $B(n)$ denote the closed ball of radius n in $\mathbb{R}^d$. For all $\mu \in \Delta^1(\mathbb{R}^d)$ we define the Borel measurable function

$$n^*(\mu) = min\{n \in \mathbb{N}^* : \int_{B(n)^c} |x|_1 d\mu(x) \leq \eta\}$$

For each $B(n)$, let $(\Omega^n_q)_{q=1,..,Q_n}$ be a finite measurable partition of $B(n)$ of mesh smaller than η and $(x^n_q)_{q=1,..Q_n}$ a sequence of points such that $x^n_q \in \Omega^n_q$ for all q. Define then Λ_n as the subset of convex combinations $\lambda \in \Delta(Q_n)$ such that for all $q = 1, .., Q_n$, $\lambda_q \in \{\frac{k}{N_n},\ k = 0, .., N_n\}$ for some N_n to be fixed later. Let us define the following applications, R being the conditional law of μ on $B(n^*(\mu))$ and λ^n a discretization of this law on a grid.

$$R\,:\,\Delta^1(\mathbb{R}^d)\to\Delta^1(\mathbb{R}^d)\,:\,\mu\to R(\mu)$$

$$\text{where for any Borel subset } U,\quad R(\mu)(U)=\frac{\mu(U\cap B(n^*(\mu)))}{\mu(B(n^*(\mu)))}$$

$$\lambda^n\,:\,\Delta(B(n))\to\Lambda_n\,:\,\mu\to\begin{cases}\lambda^n_q(\mu)=\frac{\lfloor R(\mu)(\Omega^n_q)N_n\rfloor}{N_n} & \text{for}\quad q=1,...,Q_n-1\\ \lambda^n_{Q_n}(\mu)=1-\sum_{q=1}^{Q_n-1}\lambda^n_q(\mu) & \end{cases}$$

Hereafter, we adopt the convention that $n=n^*(\mu)$ in order to shorten notations. At first, these applications allow to define an approximation of μ by the probability with finite support $\sum_{q=1}^{Q_n}\lambda^n_q(\mu)\delta_{x^n_q}$. Indeed, we have the following inequalities (we leave the proof to the reader)

$$d_{W_1}(\mu,R(\mu))\le 2\eta$$

$$d_{W_1}(R(\mu),\sum_{q=1}^{Q_n}R(\mu)(\Omega^n_q)\delta_{x^n_q})\le\eta$$

$$d_{W_1}(\sum_{q=1}^{Q_n}\mu(\Omega^n_q)\delta_{x^n_q},\sum_{q=1}^{Q_n}\lambda^n_q(\mu)\delta_{x^n_q})\le\frac{2nQ_n}{N_n}$$

Now, for each element $\lambda\in\Lambda_n$, there exists an η-optimal strategy of P1 in the game $\Gamma_1(\sum_{q=1}^{Q_n}\lambda_q\delta_{x^n_q})$ denoted $h_n(\lambda)$, which we can clearly identify to an element of $\Delta(I)^{Q_n}$ with coordinates $h_n(\lambda)_q$. Let also $i_0\in I$ be a fixed element. We are now able to define P1's strategy:

$$H_\varepsilon(L,\mu)=\begin{cases}\delta_{i_0} & \text{if}\quad L\notin B(n)\\ h_n(\lambda^n(\mu))_q & \text{if}\quad L\in\Omega^n_q\end{cases}$$

This defines a jointly measurable application and it remains to prove that it guarantees the right quantity. Let us fix $\mu\in\Delta^1(\mathbb{R}^d)$ and a strategy τ for P2. The triplet $(\mu,H_\epsilon(.,\mu),\tau)$ defines a joint probability π. The associated payoff is

$$\mathbb{E}_\pi[\langle L,T(i,j)\rangle]=\mathbb{E}_\pi[\langle L,T(i,j)\rangle\mathbb{1}_{L\notin B(n)}]+\mathbb{E}_\pi[\langle L,T(i,j)\rangle\mathbb{1}_{L\in B(n)}]$$

The first term is bounded by $C_T\eta$, and the second is equal to

$$\mu(B(n))\mathbb{E}_\pi[\langle L,T(i,j)\rangle\mid L\in B(n)]$$

By construction $\mu(B(n)) \geq 1 - \frac{\eta}{n}$ and we have

$$\begin{aligned}
\mathbb{E}_\pi[\langle L, T(i,j)\rangle \mid L \in B(n)] &\geq \mathbb{E}_\pi[\sum_{q=1}^{Q_n} \mathbb{I}_{L\in\Omega_q^n}\langle x_q^n, T(i,j)\rangle \mid L \in B(n)] - C_T\eta \\
&= \sum_{q=1}^{Q_n} R(\mu)(\Omega_q^n)\langle x_q^n, \int T(i,j) d[h_n(\lambda^n(\mu))_q(i)\otimes\tau(j)]\rangle - C_T\eta \\
&\geq \sum_{q=1}^{Q_n} \lambda_q^n(\mu)\langle x_q^n, \int T(i,j) d[h_n(\lambda^n(\mu))_q(i)\otimes\tau(j)]\rangle - \frac{2Q_n}{N_n}C_T n - C_T\eta \\
&\geq \underline{V}_1(\sum_{q=1}^{Q_n} \lambda_q^n(\mu)\delta_{x_q^n}) - \eta - \frac{2Q_n}{N_n}C_T n - C_T\eta
\end{aligned}$$

Defining N_n sufficiently large so that $\frac{2Q_n}{N_n}C_T n$ is bounded by η for all n, we only need to bound the difference

$$|\underline{V}_1(\sum_{q=1}^{Q_n} \lambda_q^n(\mu)\delta_{x_q^n}) - \underline{V}_1(\mu)|$$

Using that $\underline{V}_1$ is C_T-Lipschitz for the Wasserstein distance of order 1 d_{W_1} and the preceding inequalities, this quantity is bounded by $C_T(3\eta + \frac{2nQ_n}{N_n})$. Finally

$$\mathbb{E}_\pi[\langle L, T(i,j)\rangle] \geq (1-\frac{\eta}{n})(\underline{V}_1(\mu) - 4C_T(\eta + \frac{nQ_n}{N_n}) - \eta) - C_T\eta \ = \underline{V}_1(\mu) + r(\eta)$$

where $r(\eta)$ is going to zero with η. Indeed, the only term we still need to bound is $\frac{\eta}{n}\underline{V}_1(\mu)$ and this bound is given by (recall that n depends on μ)

$$\begin{aligned}
|\frac{\eta}{n}\underline{V}_1(\mu)| &\leq \frac{\eta}{n}(|\underline{V}_1(\delta_0)| + C_T\int_{\mathbb{R}^d}|x|_1 d\mu(x)) \leq \frac{\eta}{n}(|\underline{V}_1(\delta_0)| + C_T(n\mu(B_n) + \eta)) \\
&\leq \frac{\eta}{n}(|\underline{V}_1(\delta_0)| + C_T(n+1)) \leq \eta(|\underline{V}_1(\delta_0)| + 2C_T)
\end{aligned}$$

□

Proof of lemma 1.3. The proof is just sketched since the technical details are the same as for proposition 1.2. Define $\Pi(\mu,\sigma,\tau)$, the filtration $(\mathcal{F}_k)_{k=0,..,n}$ and the martingale of expected beliefs as in (1.1). With our assumptions, for all $k = 1,..,n$, after round k, the expected conditional payoff X_k given $\mathcal{F}_k$ of the $n-q$ remaining rounds is at least $\underline{V}_n(L \mid \mathcal{F}_k)$. Otherwise P1's strategy wouldn't be optimal since the expected payoff of the first k rounds cannot be greater that $\underline{V}_k([L_k])$ and using equality **??**. Suppose then that the event $\{u_k > v_k\}$ occurs with positive probability with

$$u_k = \mathbb{E}_{\Pi(\mu,\sigma,\tau)}[\langle L_k, T(i_k,j_k)\rangle \mid \mathcal{F}_{k-1}], \text{ and } v_k = \underline{V}_1([L_k \mid \mathcal{F}_{k-1}])$$

At round k, since P2 can compute the above quantities, he could deviate using an ε-optimal response to τ_q the strategy of P1 in the one round game $\Gamma_1([L_k \mid \mathcal{F}_{k-1}])$ (constructed as in the previous proof) and follow then by playing an ε-optimal response to the remaining part of the strategy of P1 in the game $\Gamma_{n-k}([L \mid \mathcal{F}_k])$ (again, this law does not depend on τ_q). The payoff of P1 in the remaining part of the game denoted Y_k will therefore be less or equal to $\underline{V}_n(L \mid \mathcal{F}_k) + \eta$ and therefore

$$v_k + \eta + Y_k \leq v_k + \underline{V}_n(L \mid \mathcal{F}_k) + 2\eta < u_k + X_k$$

for η small enough, which implies that we have constructed a profitable deviation for P2. The event $\{v_k < u_k\}$ has therefore zero probability and it follows by summation that

$$\mathbb{E}_{\Pi(\mu,\sigma,\tau)}[\sum_{k=1}^{n} \underline{V}_1([L_k \mid \mathcal{F}_{k-1}])] \geq \mathbb{E}_{\Pi(\mu,\sigma,\tau)}[\sum_{k=1}^{n} \langle L_k, T(i_k, j_k)\rangle \mid \mathcal{F}_{k-1}] = \overline{\mathcal{V}}_n(\mu)$$

□

5.1. Jensen Inequality. Let $p \in [1, \infty)$ and $\Delta^p(\mathbb{R}^d)$ the set of probabilities with finite moment of order p. The vector space M^p of finite signed Borel measures μ on $\mathbb{R}^d$ such that $\int_{\mathbb{R}^d} |x|^p d|\mu| < \infty$ is endowed with initial topology generated by the set $C_p(\mathbb{R}^d)$ of continuous functions on $\mathbb{R}^d$ with at most polynomial growth of order p. The induced topology on Δ^p is metrizable by the wasserstein distance d_{W_p} (see appendix).

Lemma 5.1: *Let $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space, $\mathcal{G} \subset \mathcal{F}$ two sub σ-algebra of $\mathcal{A}$, and f a concave upper semi-continuous mapping from Δ^p to $\mathbb{R}$ which is bounded by $C(1 + d_{W_p}(\delta_0, .))$. Then, for all $\mathbb{R}^d$-valued random variable X with finite moment of order p*
- $f([X]) \geq \mathbb{E}[f([X \mid \mathcal{F}])]$
- $f([X \mid \mathcal{G}]) \geq \mathbb{E}[f([X \mid \mathcal{F}]) \mid \mathcal{G}]$ *almost surely.*

Proof. Note that all the expectations in the proof are well-defined using the bound on f and the integrability condition on X. Since X has a finite moment of order p, we can assume that the random variable $[X \mid \mathcal{F}]$ is Δ^p-valued. Let Φ denote its distribution (in $\Delta(\Delta^p(\mathbb{R}^d))$). f being upper semi-continuous, it is sufficient to prove that $\mu = [X]$ is the barycenter of Φ. But, for all $h \in C_p(\mathbb{R}^d)$, it follows from the properties of the conditional expectation that

$$\int \langle h, \nu\rangle d\Phi(\nu) = \mathbb{E}[\mathbb{E}[h(X) \mid \mathcal{F}]] = \mathbb{E}[h(X)] = \langle h, \mu\rangle$$

which proves the first result. The second one follows by the same method. It is sufficient to prove that $[X \mid \mathcal{G}]$ is almost surely the barycenter the $\Delta(\Delta^p(\mathbb{R}^d))$-valued $\mathcal{G}$-measurable random variable

$$\Psi = [[X \mid \mathcal{F}] \mid \mathcal{G}] .$$

Applying the previous argument to a well-chosen countable subset $\mathcal{C}_0$ of $C_p(\mathbb{R}^d)$ and by using the definitions of conditional laws and conditional expectations, we have with probability one

$$\forall h \in \mathcal{C}_0, \quad \int \langle h, \nu\rangle d\Psi(\nu) = \mathbb{E}[h(X) \mid \mathcal{G}] \tag{5.1}$$

Now $\mathcal{C}_0$ can be taken as the union of $x \to (1 + |x|^p)$ and of a countable dense subset of the set of bounded uniformly continuous functions on $\mathbb{R}^d$ with respect to the metric $|.| \wedge 1$. The property (5.1) can therefore be extended to all $h \in C_p(\mathbb{R}^d)$ and this implies

$$f([X \mid \mathcal{G}]) \geq \mathbb{E}[f([X \mid \mathcal{F}]) \mid \mathcal{G}]$$

with probability one. □

REMARK 5.1: *The same results holds with the same proof when replacing Δ^p by $\Delta(P)$ for some compact convex subset P of $\mathbb{R}^d$ and considering a bounded concave u.s.c. function and P-valued random variables.*

CHAPITRE 2

Problèmes de variation maximale de martingales.

On présente dans ce chapitre une généralisation des résultats de Mertens-Zamir [**43**] et De Meyer [**25**] sur une famille de problèmes de variation maximale de martingales. Les sections 3 à 6 concernent la convergence du problème en temps discret vers un problème limite d'optimisation sur un ensemble de lois de martingales en temps continu.

Les sections 7 et 8 s'intéressent à la représentation duale en termes d'équations aux dérivées partielles de ce problème limite.

Enfin, les sections 9 à 11 concernent le comportement asymptotique des maximiseurs du problème en temps discret et la caractérisation des solutions du problème limite. On retrouve en particulier les résultats asymptotiques obtenus dans [**25**].

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1. Introduction

We call problems of maximal variation of martingales of length $n \in \mathbb{N}$ a class of stochastic optimization problems generalizing the problem of maximal L^1-variation introduced in [**43**] to more general functions than the L^1-norm. Given some real-valued function V and a probability μ over $\mathbb{R}^d$, we aim to maximize a functional called the V-variation, over the set $\mathcal{M}_n(\mu)$ of $\mathbb{R}^d$-valued martingales of length n whose terminal distribution is Blackwell dominated by μ (see definition 12.1). More precisely, let V be a real-valued function defined on the set of probabilities over $\mathbb{R}^d$. We define the V-variation of length n of the martingale $(L_k)_{k=1,..,n}$ as

$$\mathcal{V}_n^V((L_k)_{k=1,..,n}) = \mathbb{E}[\sum_{k=1}^{n} V([L_k - L_{k-1} \mid (L_i, i \leq k-1)])]$$

where $[L_k - L_{k-1} \mid (L_i, i \leq k-1)]$ denotes the conditional law of $L_k - L_{k-1}$ given $(L_i, i \leq k-1)$ with the convention $L_0 = \mathbb{E}[L_1]$. The value function of the above problem is denoted

$$V_n(\mu) = \frac{1}{\sqrt{n}} \sup_{(L_k)_{k=1,..,n} \in \mathcal{M}_n(\mu)} \mathcal{V}_n^V((L_k)_{k=1,..,n})$$

This problem has been recently studied in De Meyer [**25**] for the case $d = 1$. The main result in [**25**] is two-fold. At first, a characterization of the limit $V_\infty = \lim_n V_n$ as a maximal covariance function is given. Then, it is shown that any sequence of asymptotically optimal martingales for V_n, considered as piecewise constant continuous-time processes, converges in law to a specific continuous-time martingale called Continuous Martingale of Maximal Variation (CMMV) when n goes to ∞. The most surprising aspect of this result is that the law of the limit process CMMV does not depend on V, and neither does V_∞ up to a multiplicative constant.

Our aim of this work is to extend these two results in a multi-dimensional setting.

Assumptions on V. We introduce five assumptions denoted A1-A5 on the function V. Assumptions A1-A4 are the natural generalizations of the assumptions given in [**25**], while A5 is specific to the multi-dimensional case.

Let Δ^2 denote the set of probabilities with finite second-order moments over $\mathbb{R}^d$ and Δ_0^2 the subset of centered probabilities. Let L^2 denote the space of $\mathbb{R}^d$-valued square-integrable random variables defined on some atomless probability space. We assume that the real-valued function V defined on the set Δ_0^2 has the following properties

(A1) $V \geq 0$ and has no degenerate directions : $\forall x \in \mathbb{R}^d$, there exists $\mu \in \Delta_0^2$ such that

$$\mu(\mathbb{R}x) = 1, \text{ and } \quad V(\mu) > 0.$$

(A2) V is K-Lipschitz for the Wasserstein distance [1] of order p for some $p \in [1, 2)$.

(A3) V is positively 1-homogenous : for any random variable $X \in L^2$ and $\lambda > 0$,

$$V([\lambda X]) = \lambda V([X])$$

where $[X]$ denotes the law of X.

(A4) V is concave on Δ_0^2 (seen as a convex subset of the space of Radon measures on $\mathbb{R}^d$).

1. We will assume without loss of generality in the proofs that $1 < p < 2$.

The last assumption requires the introduction of two auxiliary functions R and r depending on V. The function R depends only on the covariance matrices of the probabilities in Δ_0^2 (denoted $cov(\mu)$). Precisely, R is defined on Δ_0^2 by

$$R(\mu) = \sup_{\nu\in\Delta_0^2 : cov(\nu)=cov(\mu)} V(\nu) = r(cov(\mu))$$

and r is the induced function defined on the set of semi-definite positive symmetric matrices S_+^d. Note that R defines naturally a function on L^2 by $R(Y) = r(cov(Y))$. Our last assumption is

(A5) R is quasiconvex on $L^2 \;\Leftrightarrow \forall \alpha \in \mathbb{R},\ \{Y \in L^2 \mid r(cov(Y)) \leq \alpha\}$ is convex in L^2.

The second formulation is a geometric constraint on the sub-level sets of r that will be called L^2-convexity (see section 3 for a discussion).

Remark 1.1: *Note that the function R is concave on Δ_0^2 and convex on L^2 (from A3 and A5), but that that the linear structures are not the same. If $d = 1$, it is easy to check that A5 is always true and that r is determined uniquely up to a multiplicative constant (see section 11).*

A simple example of such a function V is given by the L^p-norm $V(\mu) = \|X\|_{L^p}$ for some $p \in [1, 2)$ where $[X] = \mu$. A larger class of functions is obtained by considering the upper envelopes of maximal covariance functions (see section 12.1)

$$V : \mu \to \sup_{\nu\in I} C(\mu, \nu) \tag{1.1}$$

where $I \subset \Delta_0^2$ is convex, has uniformly bounded moments of order q for some $q > 2$, and contains some ν such that $cov(\nu)$ in non-degenerate. The function C is defined by

$$C(\mu, \nu) = \sup_{[X]=\mu\, ,\, [Y]=\nu} \mathbb{E}[\langle X, Y\rangle]. \tag{1.2}$$

where $\langle ., .\rangle$ denotes the scalar product in $\mathbb{R}^d$, and the maximum is over all the joint distributions of pairs (X, Y) meeting the marginal constraints $[X] = \mu$ and $[Y] = \nu$.

Main results. In order to state the first result, we need the following definition

Definition 1.1: *Given the function r defined above, the subsets Γ and G of S_+^d are defined by*

$$\Gamma = co(G),\ \textit{and}\quad G = \{P \in S_+^d\ :\ \forall M \in \mathbb{M}_d, r(MM^T) \leq 1 \Rightarrow Tr(\sqrt{P}M) \leq 1\}$$

where $\mathbb{M}_d$ denotes the set of $d \times d$ matrices, $Tr()$ the trace, and $\sqrt{P}$ the semi-definite positive square root of P.

Our first main theorem shows that $\lim_n V_n$ depends only on V through the auxiliary function R (or r).

Theorem 1.1: *Under assumptions A1-A5, the limit V_∞ of the sequence V_n exists and is given by*

$$\lim_{n\to\infty} V_n(\mu) = V_\infty(\mu) \triangleq \max_{\nu\in Q_\Gamma(1)} C(\mu, \nu)$$

where Q_Γ is the compact convex set of laws of RR^d-valued martingales $(Z_t)_{t\in[0,1]}$ with continuous trajectories whose quadratic covariation process $\langle Z\rangle_t$ is such that with probability 1

$$\forall\, 0 \leq s < t \leq 1, \qquad \frac{1}{t-s}(\langle Z\rangle_t - \langle Z\rangle_s) \in \Gamma \tag{1.3}$$

and $Q_\Gamma(1)$ is the set of laws of variables Z_1 for all the processes Z whose law is in Q_Γ.

The proof of this theorem has two distinct parts. The first one shows that the limit V_∞ is in fact an upper bound for $limsup_n\, V_n$, and relies on a central limit result for martingales (see proposition 5.1). The second part shows that V_∞ is a lower bound for $liminf_n\, V_n$. This lower bound property relies on the following reformulation of the problem V_∞ as an optimization problem over continuous-time martingales distributions (lemma 6.2)

$$V_\infty(\mu) = W_{ac}(\mu) \triangleq \underset{X\in\mathcal{M}_{ac}(\preceq_\mu)}{sup} \mathbb{E}[\int_0^1 r(\frac{d}{ds}\langle X\rangle_s)ds] \tag{1.4}$$

where $\mathcal{M}_{ac}(\preceq_\mu)$ is the set of distributions of martingales $(X_t)_{t\in[0,1]}$ with continuous trajectories, having a quadratic variation process $(\langle X\rangle_t)_{t\in[0,1]}$ which is absolutely continuous with respect to the Lebesgue's measure, and whose final distribution $[X_1]$ is Blackwell dominated by μ. This continuous-time formulation allows us to prove in proposition 6.1 that for an ε-optimal $X \in \mathcal{M}_{ac}(\preceq_\mu)$, there exists a sequence of discretizations $X^n = (X^n_k)_{k=1,..,n}$ of X that are asymptotically ε-optimal for V_n (i.e. such that $liminf_n\, V_n(X^n) \geq W_{ac}(X) - \varepsilon$). We emphasize that our approximation procedure is not the usual time-discretization, since we have to introduce a second level of discretization based on the central limit theorem for the Wasserstein distance.

In the second part, we introduce the convex dual problem V^*_∞ of V_∞, and obtain a dual equality using results appearing in the theory of optimal transport (proposition 7.1). This dual problem is then shown to be a PDE problem of HJB type appearing in stochastic control theory (proposition 8.1).

The third part of this work is devoted to the problem of identifying the limits of optimizers of V_n. Precisely, given a discrete-time process $(L_1, .., L_n)$, the continuous-time version of this process is defined by

$$\Pi^n_t = L_{\lfloor nt\rfloor} \quad \text{for } t \in [0,1]$$

where $\lfloor a\rfloor$ is the greatest integer less or equal to a. We aim to characterize the limits in law of the continuous-time versions of asymptotically optimal sequences in $\mathcal{M}_n(\mu)$ for the problem $V_n(\mu)$. This is done in two steps. The first one is the following reformulation of V_∞

$$V_\infty(\mu) = W(\mu) \triangleq \underset{X\in\mathcal{M}(\preceq_\mu)}{max} H(X)$$

where $\mathcal{M}(\preceq_\mu)$ is the set of distributions of martingales $(X_t)_{t\in[0,1]}$ with càdlàg trajectories [2] whose final distribution is Blackwell dominated by μ. The functional H is defined in section 9 and extends the integral functional given in (1.4) to the set $\mathcal{M}(\preceq_\mu)$.

This second formulation is introduced in order to obtain compactness, and to show that the set of maximizers of W contains the set of accumulation points of the maximizers of the discrete-time problem. Precisely

2. Recall that càdlàg is the french acronym for right-continuous with left-hand limits

Theorem 1.2: *Let (L^n) be an asymptotically maximizing sequence of $V_n(\mu)$ in $\mathcal{M}_n(\mu)$. Then the continuous-time versions of these martingales define a weakly relatively compact sequence of laws for the Meyer-Zheng topology (see* [**45**]*) and any limit point belongs to*

$$\mathcal{P}_\infty(\mu) = \underset{X\in\mathcal{M}(\preceq_\mu)}{argmax}\ H(X).$$

In the second step, we use the dual formulation to derive some properties of $\mathcal{P}_\infty$. It will be too long to state completely these results in the introduction, but as a motivation and in order to give an insight of the characterization we obtain, let us mention a corollary of theorem 10.1, theorem which also implies the former results obtained in [**25**] concerning the class CMMV (see section 11).

Theorem 1.3: *Let $u(t,x)$ be the unique viscosity solution (see lemma 8.7) of the following HJB equation*

$$\begin{cases} -\partial_t u - \sup_{P\in\Gamma} Tr(P\nabla^2 u) & = & 0 & in & (0,1)\times\mathbb{R}^d \\ u(1,x) & = & f(x) & in & \mathbb{R}^d \end{cases} \tag{1.5}$$

where f is a C^1 convex function on $\mathbb{R}^d$ such that ∇f has at most polynomial growth. Assume that u is a classical $C^{1,2}$ solution. Let Z be a martingale whose law $\mathbb{P}$ is in Q_Γ and such that

$$\frac{d}{dt}\langle Z\rangle_t \in \underset{P\in\Gamma}{argmax}\ Tr(P\nabla^2 u(t,Z_t))\ dt\otimes d\mathbb{P}\ almost\ surely \tag{1.6}$$

Then, if μ is the law of $\nabla f(Z_1)$, the set $\mathcal{P}_\infty(\mu)$ is exactly the set of laws of the martingales

$$(X_t)_{t\in[0,1]} = (\nabla u(t,\widetilde{Z}_t))_{t\in[0,1]}$$

where the law of $\widetilde{Z}$ ranges through all the laws in Q_Γ verifying (1.6) *and $\nabla f(\widetilde{Z}_1)\sim\mu$.*

2. Problem and Notations

2.1. Equivalent Formulations of the problem. As it will be convenient to consider martingales defined with respect to a larger filtration than the filtration generated by the process itself, let us now introduce an equivalent formulation of the V-variation.

Definition 2.1: *$\mathfrak{M}_n(\mu)$ is the collection of martingales $(L_k,\mathcal{F}_k)_{k=1,..,n}$ defined of some filtered probability space $(\Omega,\mathcal{A},(\mathcal{F}_k)_{k=1,..,n},\mathbb{P})$, of length n and whose final distribution is Blackwell dominated by μ $([L_n]\preceq\mu)$. By convention, we set $\mathcal{F}_0=\{\Omega,\emptyset\}$.*

With a slight abuse of notations, we extend the definition of the V variation to martingales in $\mathfrak{M}_n(\mu)$ by

$$\mathcal{V}_n^V((L_k,\mathcal{F}_k)_{k=1,..,n}) = \mathbb{E}[\sum_{k=1}^n V([L_k - L_{k-1}\mid\mathcal{F}_{k-1}])] \tag{2.1}$$

Lemma 2.1:

$$V_n(\mu) = \frac{1}{\sqrt{n}} \sup_{((L_k,\mathcal{F}_k)_{k=1,..,n})\in\mathfrak{M}_n(\mu)} \mathcal{V}_n^V((L_k,\mathcal{F}_k)_{k=1,..,n}) \tag{2.2}$$

PROOF. Given a distribution $[(L_1,..,L_n)] \in \mathcal{M}_n(\mu)$, then the two notions of V-variation agree if we define $(\mathcal{F}_k^L)_{k=1,..,n}$ as the natural filtration of $(L_1,..,L_n)$, i.e.

$$\mathcal{F}_k^L = \sigma(L_1,..,L_k) \text{ for } k=1,..,n \text{ and } \mathcal{F}_0 = \{(\mathbb{R}^d)^n, \emptyset\}$$

then

$$\mathcal{V}_n^V((L_k)_{k=1,..,n}) = \mathcal{V}_n^V((L_k,\mathcal{F}_k^L)_{k=1,..,n})$$

This proves that V_n is not greater than the right-hand side of (2.2). To prove the reverse inequality, let $((L_k,\mathcal{F}_k)_{k=1,..,n}) \in \mathfrak{M}_n(\mu)$. Since V is concave and d_{W_p}-Lipschitz, it follows from Jensen's inequality (lemma 5.1 in chapter 1) that for all $k=1,..,n$

$$V([L_k - L_{k-1} \mid \mathcal{F}_{k-1}]) \leq V([L_k - L_{k-1} \mid \mathcal{F}_{k-1}^L])$$

The proof follows then by summation over k. □

NOTATION 5: *The function V is extended (keeping the same notation) on Δ^2 by*

$$V([X]) \triangleq V([X - \mathbb{E}[X]]) \tag{2.3}$$

The same notation will be used in the next sections with the functions R and R'. We have therefore for all martingales

$$\mathcal{V}_n^V((L_k,\mathcal{F}_k)_{k=1,..,n}) = \mathbb{E}[\sum_{k=1}^n V([L_k \mid \mathcal{F}_{k-1}])]$$

2.2. Index of Notations.

- $\Delta(X)$ denotes the set of probabilities defined on the Borel σ-field of a topological space X, endowed with the usual weak* topology.
- $|x|_p$ stands for the usual p-norm in $\mathbb{R}^d$ and we omit the index when $p=2$ (euclidian norm), the scalar product is denoted $\langle .,. \rangle$, vectors are identified to column matrices and $.^T$ denotes transposition.
- If a $\mathbb{R}^d$-valued random variable X defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ has law μ, we write $X \sim \mu$ or $[X] = \mu$. For a sub-σ-field $\mathcal{G}$, $[X \mid \mathcal{G}]$ denotes a regular version of the conditional law of X given $\mathcal{G}$. Recall that $[X \mid \mathcal{G}]$ defines a $\mathcal{G}$ measurable map with values in $\Delta(\mathbb{R}^d)$ (see appendix).
- The vectorial L^p-norm is defined by $\|X\|_{L^p} = \|\mu\|_p = \mathbb{E}[\sum_{i=1}^d |X^i|^p]^{1/p}$.
- $\mathbb{M}_d$ denotes the set of $d\times d$ square matrices and S^d the subset of symmetric matrices. S_+^d (resp. S_{++}^d) denotes the cone of symmetric semi-definite (resp. definite) positive matrices and $\leq$ the associated order between symmetric matrices. The norm $|.|$ on these sets is the induced euclidian norm of $\mathbb{M}_d$. For $M \in S_+^d$, $\sqrt{M} = M^{\frac{1}{2}}$ is the unique $N \in S_+^d$ such that $N^2 = M$.

3. Properties of the auxiliary function R

DEFINITION 3.1: *The two auxiliary functions R and r are respectively defined on Δ_0^2 and S_+^d by the relation*

$$R(\mu) = \sup_{\nu \in \Delta_0^2 : cov(\nu) = cov(\mu)} V(\nu) = r(cov(\mu))$$

which implies

$$\forall M \in S_+^d, \quad r(M) = \sup_{\nu \in \Delta_0^2 : cov(\nu) = M} V(\mu)$$

The next two lemmas and definitions explore the notion of L^2-convexity in S_+^d.

LEMMA 3.1: *For all $P, Q \in S_d^+$ and $\mu \in \Delta^2$ such that $cov(\mu) = P$, we have*

$$\sup_{\nu \in \Delta_0^2 : cov(\nu) \leq Q} C(\mu, \nu) = \sup_{\nu \in \Delta_0^2 : cov(\nu) = Q} C(\mu, \nu) = Tr\left((\sqrt{P} Q \sqrt{P})^{\frac{1}{2}}\right)$$

$$= \sup_{D \in \mathbb{M}_d \,:\, DD^T = Q} Tr(ND)$$

where the last equality holds for any N such that $NN^T = P$ (in particular $\sqrt{P}$).

PROOF. If $X \sim \mu$ and $Y \sim \nu$ are given random variables such that $U = Q - cov(\nu) \geq 0$, one can construct a variable Z independent of (X, Y) such that $\mathbb{E}[Z] = 0$ and $cov(Z) = U$. From this construction $\mathbb{E}[\langle X, Y + Z \rangle] = \mathbb{E}[\langle X, Y \rangle]$ and $cov(Y + Z) = Q$, and this proves the first equality. If X, Y are random variables in L^2 such that $X \sim \mu$ and $cov(Y) = Q$, then $Z = (X, Y)$ is a vector of random variables taking values in $\mathbb{R}^{2d}$ so that

$$cov(Z) = C_\psi = \begin{pmatrix} P & \psi \\ \psi^T & Q \end{pmatrix} \geq 0$$

and $\mathbb{E}[\langle X, Y \rangle] = \mathbb{E}[\langle X - \mathbb{E}[X], Y \rangle] = Tr(\psi)$. The value of the semi-definite positive program

$$\begin{cases} max\ Tr(\psi) \\ C_\psi \in S_+^d \end{cases}$$

is given by

$$\max_{C_\psi \in S_+^d} Tr(\psi) = Tr\left((\sqrt{P} Q \sqrt{P})^{\frac{1}{2}}\right)$$

for any pair of covariance matrices (see [**49**] th. 3.4.1 and references therein), and this bound is reached by compactness. To prove the second equality, if $P = cov(X) \in S_{++}^d$ consider the mapping $x \in \mathbb{R}^d \to Mx \in \mathbb{R}^d$ with the matrix M defined by

$$M = \sqrt{P}^{-1} \left(\sqrt{P} Q \sqrt{P}\right)^{\frac{1}{2}} \sqrt{P}^{-1}$$

Then with $Y = M(X - \mathbb{E}[X])$ we have $\mathbb{E}[\langle X, Y \rangle] = Tr\left((\sqrt{P} Q \sqrt{P})^{\frac{1}{2}}\right)$. For the last equality, one can always construct a random variable U such that $\mathbb{E}[U] = 0$ and $cov(U) = I_d$ and $X = NU + \mathbb{E}[X] \sim \mu$, then with $Y = DU$, we have $\mathbb{E}[\langle X, Y \rangle] = Tr(DN)$ which implies the result since this bound is reached with $D = MN$. These equalities extend then to S_+^d by a continuity argument. □

NOTATION 6:

- *L^2 denotes the hilbert space $L^2([0,1], dx; \mathbb{R}^d)$ of $\mathbb{R}^d$-valued random variables.*
- *L_0^2 denotes the subset of random variables $X \in L^2$ such that $\mathbb{E}[X] = 0$.*
- *A function f defined on Δ^2 is "extended" to L^2 using the convention $f(X) = f([X])$.*
- *To a subset Λ of Δ^2, we associate the subset $\widetilde{\Lambda}$ of L^2 containing elements of L^2 whose distributions belong to Λ*

– *To a subset Θ of S^d_+, we associate the subset $\widehat{\Theta}$ of variables $X \in L^2_0$ such that*

$$cov(X) = \mathbb{E}[XX^T] \in \Theta.$$

Definition 3.2: *A subset Θ of S^d_+ is said to be L^2-convex if $\widehat{\Theta}$ is convex in L^2_0.*

Definition 3.3: *The polar set C° of $C \subset L^2_0$ is defined by*

$$C^\circ = \{X \in L^2_0 \,:\, \sup_{Y \in C} \mathbb{E}[\langle X, Y\rangle] \leq 1\}$$

Now we give a characterization of L^2-convexity.

Lemma 3.2: *Let Θ be a nonempty comprehensive subset od S^d_+, i.e. such that*

$$P \in \Theta \text{ and } Q \leq P \ \Rightarrow\ Q \in \Theta.$$

Then $\widehat{\Theta}^\circ = \widehat{\Theta^\dagger}$ where

$$\Theta^\dagger = \{P \in S^d_+ \,:\, \sup_{Q \in \Theta} Tr\left((\sqrt{P}Q\sqrt{P})^{\frac{1}{2}}\right) \leq 1\}$$

and $\widehat{\Theta}$ is closed and convex in L^2_0 if and only if $\Theta^{\dagger\dagger} = \Theta$.

Proof. Note that $\widehat{\Theta} = \cup_{Q\in\Theta}\{Y \in L^2_0 \,:\, cov(Y) \leq Q\}$. We claim that if $X \in L^2_0$ is μ-distributed, then

$$\sup_{Y \in L^2_0 \,:\, cov(Y) \leq Q} \mathbb{E}[\langle X, Y\rangle] = \sup_{\nu \in \Delta^2_0 : cov(\nu) \leq Q} C(\mu, \nu)$$

Given a joint law $\pi \in \mathcal{P}(\mu, \nu)$ such that $cov(\nu) \leq Q$, then enlarging the probability space, we can define a variable Y such that $(X, Y) \sim \pi$. But we can replace Y by its conditional expectation given X, $\mathbb{E}[Y \mid X] = \phi(X)$, which is defined on the original probability space as a function of X. We check easily that $\mathbb{E}[\langle X, Y\rangle] = \mathbb{E}[\langle X, \phi(X)\rangle]$ and that $cov(\phi(X) \leq cov(Y) \leq Q$. It follows then from lemma 3.1 that with $cov(X) = P$ we have

$$\sup_{Y \in \widehat{\Theta}} \mathbb{E}[\langle X, Y\rangle] = \sup_{Q \in \Theta} \sup_{Y \in L^2_0 \,:\, cov(Y) \leq Q} \mathbb{E}[\langle X, Y\rangle] = \sup_{Q \in \Theta} Tr\left((\sqrt{P}Q\sqrt{P})^{\frac{1}{2}}\right)$$

The conclusion follows then from the usual characterization of the closed convex hull as the bipolar in L^2_0 since our assumption implies that $0 \in \Theta$. □

The following lemma lists some properties of the function r.

Lemma 3.3: *The function r is non-negative, concave, non-decreasing and continuous on S^d_+. Moreover:*

$$\forall \lambda > 0, \qquad r(\lambda M) = \sqrt{\lambda} r(M). \tag{3.1}$$

$$r(M) = \max_{\mu \in \Delta^2_0 : cov(\mu) \leq M} V(\mu) \tag{3.2}$$

Proof. Note at first that the d_{W_p}-closure of $\{\nu \in \Delta^2_0 \,:\, cov(\nu) = M\}$ is $\{\nu \in \Delta^2_0 \,:\, cov(\nu) \leq M\}$ (see the proof of lemma 6.5 in section 12), so that (3.2) follows from A2. Since cov is linear and V is 1-homogenous, non-negative and concave, the non-negativeness, concavity and (3.1) are obvious. Let us prove continuity on some compact subset E. Note that the subset $\{\mu \in \Delta^2_0 : cov(\mu) \leq M\}$ is d_{W_p}-compact since moments of order $2 > p$ are uniformly bounded.

The continuity of r follows therefore from Berge's maximum theorem (see [**8**] p.116) since the set-valued mapping

$$M \to \{\mu \in \Delta_0^2 : cov(\mu) \leq M\}$$

is both upper and lower semi-continuous when Δ_0^2 is endowed with the metric d_{W_p}. $\square$

Let $F = \{M \in S_+^d \ : \ r(M) \leq 1\}$. Using the notations of the previous lemmas, the sets Γ and G defined in (1.1) are such that

$$\Gamma = co(G) \text{ and } \quad G = F^\dagger = \{P \in S_+^d \ : \ \sup_{M \in \mathbb{M}_d \, : \, MM^T \in F} Tr(M\sqrt{P}) \leq 1\}$$

Our main result in this section is the following upper bound for V, which is a modification of R that takes into account the Lipschitz assumption A2.

Proposition 3.1: *$V \leq R'$ on Δ_0^2 with $R'(\mu) = \sup_{\nu \in T} C(\mu, \nu)$ and*

$$T = \{\nu \in \Delta_0^2 \ : \ cov(\mu) \in \Gamma \, , \ \|\mu\|_{p'} \leq 2K\}$$

with p' the conjugate exponent of p.

Proof. Since these two functions also positively homogenous (A3), it is sufficient to prove the inclusion

$$\{X \in L_0^2 \mid R'(X) \leq 1\} \subset \{X \in L_0^2 \mid V(X) \leq 1\} \tag{3.3}$$

Note at first that lemma 3.3 and A1 imply that F is compact and is a neighborhood of 0 in S_+^d. Define the following subsets of Δ_0^2

$$\overline{F} = \{\mu \in \Delta_0^2 \ : \ r(cov(\mu)) \leq 1\} \qquad B_{1/K}^p = \{\mu \in \Delta_0^2 \ : \ \|\mu\|_p \leq \frac{1}{K}\}$$

We claim that

(3.4) $\qquad co(\widehat{F}, \widetilde{B}_{1/K}^p)$ is a closed convex set and is included in $\{X \in L_0^2 \mid V(X) \leq 1\}$.

Using assumption A5, $\widehat{F}$ is weakly compact and convex and $\widetilde{B}_{1/K}^p$ is weakly closed and convex, the convex envelope is then weakly closed, hence closed in L_0^2 for the norm topology. Let $X \in \widehat{F}$, $Y \in \widetilde{B}_{1/K}^p$ and $\lambda \in [0, 1]$. Using the Lipschitz property of V, we deduce

$$\begin{aligned} V(\lambda X + (1-\lambda)Y) &\leq V(\lambda X) + K \, \|(1-\lambda)Y\|_{L^p} \\ &\leq \lambda r(cov(X)) + (1-\lambda) \leq 1 \end{aligned}$$

which proves (3.4). Next, for $X \in L_0^2$, we define the function $R^*(X) = \sup_{Y \in \widehat{F}} \mathbb{E}[\langle X, Y\rangle]$. From lemma 3.1, the function R^* depend only on $cov(X)$, that is $R^*(X) = r^*(cov(X))$ with

$$r^*(P) = \sup_{Q \in F} Tr\left((\sqrt{P}Q\sqrt{P})^{\frac{1}{2}}\right)$$

Since F is a neighborhood of $\{0\}$ in S_+^d, we can always choose $Q = \lambda P \in F$ with $\lambda > 0$ in the above supremum and it follows that r^* is positive outside $\{0\}$. r^* is continuous since F is compact. Therefore $G = \{M \in S_+^d \ : \ r^*(M) \leq 1\}$ is compact and a neighborhood of $\{0\}$ in S_+^d. Define

$$\overline{\Gamma} = \{\mu \in \Delta_0^2 \ : \ cov(\mu) \in \Gamma\} \qquad B_{2K}^{p'} = \{\mu \in \Delta_0^2 \ : \ \|\mu\|_{p'} \leq 2K\}$$

The function $R' : \Delta^2 \to \mathbb{R}$ is equal to

$$R'(\mu) = \underset{\pi\in\mathcal{P}(\mu,\overline{\Gamma}\cap B^{p'}_{2K})}{sup} \int \langle x,y\rangle d\pi(x,y) = \underset{\nu\in\overline{\Gamma}\cap B^{p'}_{2K}}{sup} C(\mu,\nu)$$

Extending the function R' on L^2, we have that :

$$\forall X \in L^2,\ R'(X) = \underset{Y\in\widehat{\Gamma}\cap\widetilde{B}^{p'}_{2K}}{sup} \mathbb{E}[\langle X,Y\rangle] \tag{3.5}$$

The proof of (3.5) proceeds as for lemma 3.2 since $\widehat{\Gamma}\cap\widetilde{B}^{p'}_{K}$ is stable by conditional expectations. Next, we prove that

$$\{X \in L^2_0 \mid R'(X) \leq 1\} \subset co(\widehat{F}, \widetilde{B}^{p}_{1/K}) \tag{3.6}$$

Suppose that $Y \notin co(\widehat{F}, \widetilde{B}^{p}_{1/K})$, then there exists Z in L^2_0 such that :

$$\mathbb{E}[\langle Z,Y\rangle] > \alpha = \underset{X\in co(\widehat{F},\widetilde{B}^{p}_{1/K})}{sup} \mathbb{E}[\langle Z,X\rangle]$$

This implies :

$$\alpha \geq \underset{X\in\widetilde{B}^{p}_{1/K}}{sup} \mathbb{E}[\langle Z,X\rangle] \geq \frac{1}{2K}\|Z\|_{p'}$$

$$\alpha \geq \underset{X\in\widehat{F}}{sup} \mathbb{E}[\langle Z,X\rangle] = R^*(Z)$$

The first inequality implies $\|Z\|_{p'} < \infty$ and the second that $\frac{Z}{\alpha} \in \widehat{G} \subset \widehat{\Gamma}$ since R^* is the support function in L^2_0 of $\widehat{F} = \widehat{G}^\circ$. Now $\frac{Z}{\alpha} \in \widehat{\Gamma}\cap\widetilde{B}^{p'}_{2K}$ and $R'(Y) \geq \mathbb{E}[\langle\frac{Z}{\alpha}, Y\rangle] > 1$ which proves (3.6). Finally, we deduce (3.3) from (3.4) and (3.6) and this concludes the proof. □

During the proof, we used the following property which follows from the duality between $\widehat{G}$ anf $\widehat{F}$ in L^2_0

Lemma 3.4:

$$\forall M \in \mathbb{M}_d, \quad r(M) = \underset{N\in\mathbb{M}_d\,:\,NN^T\in G}{max} Tr(MN)$$

4. An upper bound for the V-variation

Since we proved in the previous section that $V \leq R'$, we have

$$\mathcal{V}^V_n((L_k,\mathcal{F}_k)_{k=1,..,n}) \leq \mathcal{V}^{R'}_n((L_k,\mathcal{F}_k)_{k=1,..,n})$$

for any martingale where $\mathcal{V}^{R'}_n$ is defined by replacing V by R' in (2.1). Therefore

$$\sqrt{n}V_n(\mu) \leq \overline{\mathcal{V}}^{R'}_n(\mu) \triangleq \underset{(L_k,\mathcal{F}_k)_{k=1,..,n}\in\mathfrak{M}_n(\mu)}{sup} \mathcal{V}^{R'}_n((L_k,\mathcal{F}_k)_{k=1,..,n})$$

Definition 4.1: *Define T^n as the set of distributions in $\nu \in \Delta^2((\mathbb{R}^d)^n)$ of sequences $S_1,..S_n$ such that*

$$\forall k = 1,..,n, \qquad [S_k \mid S_1,...,S_{k-1}] \in T,\ \nu\ a.s.$$

where T is defined in proposition 3.1.

Then we have

Lemma 4.1:
$$\overline{\mathcal{V}}_n^{R'}(\mu) = \underset{[(S_k)_{k=1,..,n}]\in T^n,[L]\preceq\mu}{max} \mathbb{E}[\langle L, \sum_{k=1}^n S_k\rangle]$$

Proof. At first, note that T^n is compact convex. Indeed, by a monotone convergence argument $\nu \in T^n$ is equivalent to

$$\mathbb{E}_\nu[\phi(S_1,..,S_{k-1})S_k] = 0, \qquad \mathbb{E}_\nu[\phi(S_1,..,S_{k-1})S_kS_k^T] \in \mathbb{E}_\nu[\phi(S_1,..,S_{k-1})].\Gamma$$
$$\text{and} \qquad \mathbb{E}_\nu[\phi(S_1,..,S_{k-1})|S_k|_{p'}^{p'}] \leq (2K)^{p'}\mathbb{E}_\nu[\phi(S_1,..,S_{k-1})]$$

for all $k = 1,..,n$, where ϕ ranges through all nonnegative continuous functions bounded by 1. Since all these applications are affine and continuous, it defines a closed convex set, and relative compactness follows from the uniform bound on the moments of order p'. Existence of a maximum follows therefore from lemma 12.3 and 12.1. For any law of martingale $[(L_k)_{k=1,..,n}] \in \mathcal{M}_n(\mu)$, denoting $(\mathcal{F}_k)_{k=1,..,n}$ the natural filtration of $(L_k)_{k=1,..,n}$, we will prove

$$\mathcal{V}_n^{R'}((L_k,\mathcal{F}_k)_{k=1,..,n}) \leq \underset{S\in T^n,[L]\preceq\mu}{max} \mathbb{E}[\langle L, \sum_{k=1}^n S_k\rangle] \tag{4.1}$$

Recall that for $\kappa \in \Delta^2$, with R' defined on Δ^2 by (2.3)

$$R'(\kappa) = \underset{\nu\in T}{sup}\, C(\kappa,\nu) = \underset{\pi\in\mathcal{P}(\kappa,T)}{max} \int \langle x,y\rangle d\pi(x,y)$$

$\mathcal{P}(\kappa,T)$ is compact and from lemma 12.3 the map

$$\pi \to \int \langle x,y\rangle d\pi(x,y)$$

is continuous on $\mathcal{P}(B_r,T)$ for any $r \geq 0$ where B_r is the closed ball $\{\kappa : \|\kappa\|_2 \leq r\}$ of Δ^2. Moreover, the set-valued map

$$\kappa \to \mathcal{P}(\kappa,T)$$

has a closed graph. Therefore, using a measurable selection theorem (see appendix theorem 2.1) , the set-valued map

$$\kappa \to \underset{\pi\in\mathcal{P}(\kappa,T)}{argmax} \int \langle x,y\rangle d\pi(x,y)$$

admits a measurable selection $f(\mu)$ on B_r for any $r > 0$ and thus on Δ^2. Since the martingale has finite second order moments, the conditional second order moments are almost surely finite and there exists a family $(\mu_k)_{k=1,..,n}$ of regular versions of the conditional laws $[L_k \mid L_1,..,L_{k-1}]$ induced by $(L_k)_{k=1,..,n}$ that defines measurable maps

$$\mu_k : (\mathbb{R}^d)^{k-1} \to \Delta^2$$

such that $[L_k \mid L_1,..,L_{k-1}] = \mu_k(L_1,..,L_{k-1})$ a.s.. Up to enlarging the probability space, we assume the existence of a sequence $(V_i)_{i=1,..,n}$ of independent uniform random variables independent of $(L_1,..,L_n)$. Then we can construct [3] a sequence of random variables $(S_1,..,S_n)$ as a measurable function of $(L_k,V_k)_{k=1,..,n}$ such that the conditional laws are optimals, i.e.

$$\forall k = 1,..,n, \qquad [(L_k,S_k) \mid L_1,..,L_{k-1}] = f(\mu_k(L_1,..,L_{k-1}))\ a.s.$$

3. see theorem 1.3 and the following discussion in the appendix.

By construction, and using the martingale property

$$\mathbb{E}[\langle L_n, S_k\rangle \mid L_1,..,L_{k-1}] = \mathbb{E}[\langle L_k, S_k\rangle \mid L_1,..,L_{k-1}] = R'(\mu_k(L_1,..,L_{k-1})).$$

We deduce by summation that

$$\mathcal{V}_n^{R'}((L_k,\mathcal{F}_k)_{k=1,..,n}) = \mathbb{E}[\langle L_n, \sum_{k=1}^n S_k\rangle]$$

and inequality 4.1 follows. The other inequality is straightforward, given a pair $(L,(S_k)_{k=1,..,n})$, one has to define a martingale by projecting (using conditional expectations) L on the natural filtration of $(S_k)_{k=1,..,n}$ and the proof follows from the definition of R'. □

5. Convergence of the upper bound

Notation 7:

- $\mathbb{D}([0,1],\mathbb{R}^d)$ *denotes the set of càdlàg functions from* $[0,1]$ *to* $\mathbb{R}^d$ *endowed with the Skorokhod topology.*
- $C([0,1],\mathbb{R}^d)$ *denotes the set of continuous functions from* $[0,1]$ *to* $\mathbb{R}^d$ *endowed with the topology of uniform convergence and is identified with the subset of continuous functions in* $\mathbb{D}([0,1],\mathbb{R}^d)$.
- $\mathcal{M}$ *(resp.* $\mathcal{M}^c$*) denotes the subset of* $\Delta(\mathbb{D}([0,1],\mathbb{R}^d))$ *(resp.* $\Delta(C([0,1],\mathbb{R}^d))$*) such hat under* $\mathbb{P}\in\mathcal{M}$ *(resp.* $\mathcal{M}^c$*) the canonical coordinate process is a martingale with respect to the filtration generated by the projections.*
- *If the probability space is not mentioned, the martingales we considered will be defined on the canonical space of suitable dimension. For a martingale* $(Z_t)_{t\in[0,1]}$ *we denote* $(\mathcal{F}_t^Z)_{t\in[0,1]}$ *the right-continuous filtration it generates defined by* $\mathcal{F}_t^Z = \cap_{s>t}\sigma(Z_u, u\leq s)$ *and by* $\langle Z\rangle$ *its predictable quadratic covariation process. Filtrations are not assumed to be complete unless for the natural filtration of some Brownian motion* B *denoted* $\overline{\mathcal{F}}^B$.

Recall the following definition given in the introduction :

Definition 5.1: *Given a compact convex subset* B *of* S_+^d*, we define* Q_B *as the subset of* $\mathbb{P}$ *in* $\mathcal{M}^c$ *such that with* $\mathbb{P}$*-probability 1 :*

$$\forall 0\leq s<t\leq 1,\qquad \frac{1}{t-s}(\langle Z\rangle_t - \langle Z\rangle_s)\in B \quad \text{and} \quad Z_0=0 \tag{5.1}$$

where Z *denotes the canonical coordinate process on* $C([0,1],\mathbb{R}^d)$.

We will also need later the following notations, for $t\in[0,1]$, $Q_B(t)$ *denotes the set of laws of variables* Z_t *and* $\pi_t(Q_B)$ *the set of laws of processes* $(Z_s)_{s\leq t}$ *when the law of* $(Z_t)_{t\in[0,1]}$ *ranges through* Q_B.

Remark 5.1: *Note that this definition implies that there exists a predictable process* ρ *taking values in* B *with probability 1 such that* $\langle Z\rangle_t = \int_0^t \rho_s ds$.

Lemma 5.1: Q_B *is closed, convex and tight (hence compact) and is a face of the convex set* $\mathcal{M}^c$.

Proof. Fix $\mathbb{P} \in Q_B$, then

$$\sum_{i=1}^{d} (\langle Z^i \rangle_t - \langle Z^i \rangle_s) \le C_B(t-s)$$

with $C_B = \underset{M \in B}{sup} \|M\|$. Hence using proposition VI.3.35 and VI.4.13 in [**36**], Q_B is tight and using proposition VI.6.29 in [**36**], for any sequence $\mathbb{P}_n \in Q_B$ converging to some limit $\mathbb{P}$, we have that the sequence of distributions of $(Z^n, \langle Z^n \rangle)$ under $\mathbb{P}_n$ converges to the law of $(Z, \langle Z \rangle)$ under $\mathbb{P}$ in $\Delta(C([0,1], \mathbb{R}^d \times S_d^+))$. As a consequence the sequence of laws of $\langle Z^n \rangle$ converges to the law of $\langle Z \rangle$ so that $\mathbb{P}$ fulfills property (5.1) and thus belongs to Q_B (since the set of continuous functions verifying (5.1) is closed). To prove convexity, if $\mathbb{P} = \lambda \mathbb{P}_1 + (1-\lambda)\mathbb{P}_2$ with $\mathbb{P}_1, \mathbb{P}_2 \in Q_B$, then for $i = 1, 2$, it follows from the characterization of the quadratic covariation that

$$\forall \varepsilon > 0, \qquad \mathbb{P}_i\left(d_B\left(\frac{1}{t-s} T^n_{s,t}(Z)\right) \ge \varepsilon\right) \underset{n \to \infty}{\longrightarrow} 0 \tag{5.2}$$

where $T^n_{s,t}(Z) = \sum_{k=0}^{\infty} (Z_{(s+\frac{k+1}{n}) \wedge t} - Z_{(s+\frac{k}{n}) \wedge t})(Z_{(s+\frac{k+1}{n}) \wedge t} - Z_{(s+\frac{k}{n}) \wedge t})^T$ and $d_B(x)$ is the usual distance between x and and the compact set B. Therefore the same property holds for $\mathbb{P}$ and this clearly implies (5.1), which finally implies $\mathbb{P} \in Q_B$.

Finally, if $\mathbb{P} = \lambda \mathbb{P}_1 + (1-\lambda)\mathbb{P}_2$ with $\mathbb{P}_1, \mathbb{P}_2 \in \mathcal{M}^c$, $\lambda \in (0,1)$ and $\mathbb{P} \in Q_B$ then property (5.2) holds for $\mathbb{P}$. This property holds then also for $\mathbb{P}_1$ and $\mathbb{P}_2$, and this implies $\mathbb{P}_1, \mathbb{P}_2 \in Q_B$. $\square$

The following result is the upper bound part of theorem 1.1 :

Proposition 5.1:
$$\underset{n \to \infty}{limsup} \frac{1}{\sqrt{n}} \overline{\mathcal{V}}_n^{R'}(\mu) \le \underset{[(Z_t)_{t \in [0,1]}] \in Q_\Gamma, [L] \preceq \mu}{sup} \mathbb{E}[\langle L, Z_1 \rangle]$$

Proof. Using lemma 4.1, we have to prove that

$$\underset{n \to \infty}{limsup} \frac{1}{\sqrt{n}} \underset{[(S_k)_{k=1,..,n}] \in T^n, [L] \preceq \mu}{sup} \mathbb{E}[\langle L, \sum_{k=1}^{n} S_k \rangle] \le \underset{[(Z_t)_{t \in [0,1]}] \in Q_\Gamma, [L] \preceq \mu}{sup} \mathbb{E}[\langle L, Z_1 \rangle]$$

Let $(L^n, (S^n_k)_{k=1,..,n})$ be a maximizing sequence. Let define $\mathbb{P}_n$ as the set of processes distributions of the continuous-time processes :

$$Z^n_t = \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} S^n_k$$

The sequence $\mathbb{P}_n$ is tight since Z^n_t are martingales with respect to the right-continuous filtration $\mathcal{F}^n_t = \sigma(S^n_k,\ k \le \lfloor nt \rfloor)$ and their predictable quadratic covariation is C-tight. To prove the last point, note that $\langle Z^n \rangle_t$ is piecewise constant on the intervals $[\frac{k}{n}, \frac{k+1}{n})$ and that

$$\langle Z^n \rangle_{\frac{k+1}{n}} - \langle Z^n \rangle_{\frac{k}{n}} = \frac{1}{n} \mathbb{E}[S^n_{k+1} (S^n_{k+1})^T \mid S^n_1, ..S^n_k] \in \frac{1}{n} \Gamma \tag{5.3}$$

Since Γ is bounded by some constant C_Γ, the trace of this matrix-valued process is strongly majorized by the process $t \to \frac{C_\Gamma \lfloor nt \rfloor}{n}$. The associated sequence of laws is therefore C-tight by proposition VI.3.35 in [**36**]. To prove that the sequence $\mathbb{P}_n$ is itself C-tight, it is sufficient according to lemma VI.3.26 in [**36**] to prove that

$$\forall \varepsilon > 0,\ \mathbb{P}_n(\underset{t\in[0,1]}{sup} \mid \Delta Z^n_t \mid > \varepsilon) \underset{n\to\infty}{\longrightarrow} 0$$

where $\Delta Z^n_t = Z^n_t - Z^n_{t-}$ is the jump of Z^n at time t. Since

$$\underset{t\in[0,1]}{sup} \mid \Delta Z^n_t \mid = \underset{k=0,..,n-1}{sup} \frac{1}{\sqrt{n}} \mid S^n_{k+1} - S^n_k \mid$$

we have :

$$\begin{aligned}\mathbb{P}_n(\underset{t\in[0,1]}{sup} \mid \Delta Z^n_t \mid > \varepsilon) &\leq \sum_{k=0}^{n-1} \mathbb{P}_n(\mid S^n_{k+1} - S^n_k \mid > \varepsilon\sqrt{n}) \\ &\leq \sum_{k=0}^{n-1} \frac{\mathbb{E}^{\mathbb{P}_n}[\mid S^n_{k+1} - S^n_k \mid^{p'}]}{(\varepsilon\sqrt{n})^{p'}} \leq n\frac{(2K)^{p'}}{(\varepsilon\sqrt{n})^{p'}} \underset{n\to\infty}{\longrightarrow} 0\end{aligned}$$

Suppose now that some subsequence still denoted $\mathbb{P}_n$ converges to $\mathbb{P}$. Then the sequence of laws $\mathbb{Q}_n \in \mathbb{D}([0,1], \mathbb{R}^d \times S^d_+)$ of $(Z^n, \langle Z^n\rangle)$ is itself C-tight (corollary VI.3.33 in [**36**]) and converges to some law $\mathbb{Q}$ (up to the extraction of some subsequence) of a process (Z, A) such that Z has law $\mathbb{P}$. Now the processes Z^n and $Z^n(Z^n)^T - \langle Z^n\rangle$ are martingales with respect to $\mathcal{F}^n$ and the families of random variables $\{Z^n_t, t \in [0,1], n \in \mathbb{N}\}$ and $\{Z^n(Z^n)^T_t - \langle Z^n\rangle_t,\ t \in [0,1], n \in \mathbb{N}\}$ are uniformly integrable since respectively bounded in L^2 and $L^{\frac{p'}{2}}$. Applying proposition IX.1.12 in [**36**] to each coordinate of these process, we conclude that Z and $ZZ^T - A$ are martingales relative to the filtration $\mathcal{F}$ generated by (Z, A). The process A is $\mathcal{F}$-predictable since it is $\mathcal{F}$-adapted and has continuous trajectories. Therefore, we have with probability 1,

$$\forall t \in [0,1],\ \langle Z\rangle_t = A_t$$

This implies that for all $0 \leq s < t \leq 1$ and $\varepsilon > 0$,

$$\mathbb{P}(d_\Gamma(\frac{1}{t-s}(\langle Z\rangle_t - \langle Z\rangle_s)) > \varepsilon) \leq \liminf_n \mathbb{P}^n(d_\Gamma(\frac{1}{t-s}(\langle Z^n\rangle_t - \langle Z^n\rangle_s)) > \varepsilon) \tag{5.4}$$

and using (5.3)

$$\mathbb{P}^n(\frac{1}{t-s}(\langle Z^n\rangle_t - \langle Z^n\rangle_s) \in \frac{\lfloor nt\rfloor - \lfloor ns\rfloor}{n(t-s)}\Gamma) = 1$$

which implies

$$\mathbb{P}^n(d_\Gamma(\frac{1}{t-s}(\langle Z^n\rangle_t - \langle Z^n\rangle_s) > |1 - \frac{\lfloor nt\rfloor - \lfloor ns\rfloor}{n(t-s)}|C_\Gamma) = 0$$

This equality implies that the right-hand side of (5.4) is equal to zero for all ε, which in turn implies (5.1) and we deduce finally that $\mathbb{P} \in Q_\Gamma$.

The conclusion follows now easily, any sequence of maximizing joint distributions (L^n, Z^n_1) is tight in $\Delta(\mathbb{R}^d \times \mathbb{R}^d)$ and from the preceding discussion it converges to the law of (L, Z_1) which is in $\mathcal{P}(\{. \preceq \mu\}, Q_\Gamma(1))$. Since Z^n_1 has bounded second order moments and L^n has uniformly integrable second order moments (its marginal law is Blackwell dominated by μ), we have from lemma 12.3 that

$$\mathbb{E}[\langle L^n, Z^n_1\rangle] \underset{n\to\infty}{\longrightarrow} \mathbb{E}[\langle L, Z_1\rangle]$$

□

6. The continuous-time problem and the discretization procedure

The main result of this section is the following lower bound property that completes the proof of theorem 1.1.

PROPOSITION 6.1:

$$\liminf_{n\to\infty} V_n(\mu) \geq V_\infty(\mu)$$

This proposition will be proved using the first reformulation W_{ac} of V_∞ announced in the introduction.

DEFINITION 6.1: *Consider the canonical space $C([0,1],\mathbb{R}^d)$ endowed with the standard d dimensional Wiener measure $\mathbb{P}_0$. In order to avoid confusions, let $(B_t)_{t\in[0,1]}$ denote the canonical process and $\overline{\mathcal{F}}^B$ its natural filtration. Let $\mathcal{H}_G$ be the set of $\mathbb{M}_d$-valued $\overline{\mathcal{F}}^B$-progressively measurable processes ρ such that $\rho\rho^T \in G$. Define the subset $\widetilde{Q}_G$ of $\mathcal{M}^c$ as the set of laws of processes $Y_t = \int_0^t \rho_s dB_s$ with $\rho \in \mathcal{H}_G$.*

LEMMA 6.1: *$\widetilde{Q}_G(1)$ is dense in $Q_\Gamma(1)$.*

The proof of this lemma is postponed to section 12. This density result allows us to prove the following representation lemma.

LEMMA 6.2:

$$V_\infty(\mu) = W_{ac}(\mu) \triangleq \sup_{X\in\mathcal{M}_{ac}(\preceq_\mu)} \mathbb{E}[\int r(\frac{d}{ds}\langle X\rangle_s)ds]$$

where $\mathcal{M}_{ac}(\preceq_\mu) \subset \mathcal{M}^c$ is the subset of distributions of martingales $(X_t)_{t\in[0,1]}$ whose final distribution is Blackwell dominated by μ, and such that with probability 1, the quadratic variation process $(\langle X\rangle_t)_{t\in[0,1]}$ is absolutely continuous with respect to the Lebesgue measure. Moreover, the supremum in W_{ac} can be restricted to martingales with respect to a fixed d-dimensional Brownian filtration.

PROOF. We prove at first that $W_{ac} \leq V_\infty$. Let X be a martingale whose law is in $\mathcal{M}_{ac}(\preceq_\mu)$. Then there exists on an extension [4] denoted $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,1]},\mathbb{P})$ of our filtered probability space a d-dimensional Brownian motion W and a $\mathcal{F}$-progressively measurable process $q_s \in \mathbb{M}_d$ such that $X_t = \int_0^t q_s dW_s$ (see e.g. [**39**] theorem 3.4.2). Moreover, we have $\langle X\rangle_t = \int_0^t q_s q_s^T ds$. Define the progressively measurable process $\sigma_s = \phi(q_s)$ where ϕ is some measurable selection of the set-valued map

$$M \in \mathbb{M}_d \to \underset{N\in\mathbb{M}_d\,:\,NN^T\in G}{argmax}\ Tr(MN) \tag{6.1}$$

The law of the process $(\int_0^t \sigma_s dW_s)_{t\in[0,1]}$ is by construction in Q_Γ and we have

$$V_\infty(\mu) \geq \mathbb{E}[\langle X_1, \int_0^1 \sigma_s dW_s\rangle] = \mathbb{E}[\int_0^1 Tr(q_s\sigma_s)ds] = \mathbb{E}[\int r(\frac{d}{ds}\langle X\rangle_s)ds] \tag{6.2}$$

4. All the extensions we consider in this work are always the canonical Wiener extensions as defined in [**35**].

where the last equality follows from lemma 3.4. Let us prove the second inequality $V_\infty \leq W_{ac}$. Using lemma 6.1 and 12.3, we have

$$V_\infty(\mu) = \sup_{\nu \in \widetilde{Q}_G(1)} C(\mu, \nu)$$

Let B be a d-dimensional Brownian motion and $\overline{\mathcal{F}}^B$ its natural filtration. Using the above equality, for all $\varepsilon > 0$, there exists an ε-optimal pair $(L, (Z_t)_{t\in[0,1]})$ defined on the same probability space as B such that

$$\begin{cases} Z_t = \int_0^t \sigma_s dB_s \text{ for some } \overline{\mathcal{F}}^B \text{ progressive process } \sigma \text{ such that } \sigma_s\sigma_s^T \in G \\ [L] \preceq \mu \\ \mathbb{E}[\langle L, Z_1\rangle] \geq V_\infty(\mu) - \varepsilon \end{cases}$$

We can clearly assume that L is $\overline{\mathcal{F}}^B_1$-measurable by replacing L by its conditional expectation given $\overline{\mathcal{F}}^B_1$. Using the predictable representation property of the Brownian filtration, there exist an $\overline{\mathcal{F}}^B$ progressive process λ_s such that $L = \int_0^1 \lambda_s dB_s$. Then

$$\mathbb{E}[\langle L, Z_1\rangle] = \mathbb{E}[\int_0^1 Tr(\lambda_s\sigma_s^T)ds] \leq \mathbb{E}[\int_0^1 r(\lambda_s\lambda_s^T)ds] \leq W_{ac}(\mu)$$

which ends the proof of the second inequality and of the last assertion concerning the Brownian filtration. □

Remark 6.1: *Since we postponed the proof of density lemma 6.1, the previous lemma deserves some comments. If we replace G by Γ in* (6.1)*, the supremum will be in general strictly larger, since the problem is not convex with respect to NN^T. The proof of the density lemma basically says that after discretization in time, it is possible to replace the piecewise constant process of instantaneous covariance with values in Γ by another piecewise constant process (on a refined partition) with covariance in G (using Caratheodory's theorem). This construction cancels the effect of the possibly strict inequality we just mentioned, and we conclude using that the sequence of discretizations converge in the appropriate sense to the initial problem.*

The proof of proposition 6.1 is based on the following four technical lemmas whose proofs are standards and therefore postponed to section 12.

Some Technical Results. The first lemma is the usual central limit theorem for the Wasserstein distance. Let $RC^1(q, C) = \{\mu \in \Delta_0^2 : cov(\mu) = I_d,\ \|\mu\|_q \leq C\}$. Define then $RC^n(q, C)$ as the set of rescaled convolutions of these distributions, precisely all distributions of the variables

$$\sum_{k=1}^n \frac{S_i}{\sqrt{n}}$$

where the sequence $(S_i)_{i=1,..,n}$ is independent and identically distributed for some law $\mu \in RC^1(q, C)$. We will also use the notation $\mu^{\otimes n}$ for the law (in $\Delta((\mathbb{R}^d)^n)$) of an i.i.d. sequence of random variables of law μ.

LEMMA 6.3: *Using the previous notations and with $\mathcal{N}(0, I_d)$ being the standard centered gaussian distribution in $\mathbb{R}^d$, we have for all $q > 2$:*

$$\underset{n\to\infty}{lim} \sup_{\nu \in RC^n(q,C)} d_{W_2}(\nu, \mathcal{N}(0, I_d)) = 0$$

Moreover, given $\mu \in RC^1(q, C)$, there exists a measurable selection

$$\mu \to \pi(\mu) \in \mathcal{P}(\mu^{\otimes n}, \mathcal{N}(0, I_d))$$

such that

$$\mathbb{E}_{\pi(\mu)}[\| \sum_{k=1}^{n} \frac{S_i}{\sqrt{n}} - N\|^2] \leq \sup_{\nu \in RC^n(q,C)} d_{W_2}(\nu, \mathcal{N}(0, I_d))$$

where $((S_i)_{i=1,..,n}, N) \sim \pi(\mu)$.

The second shows an approximation result for processes without expectations (and therefore without convexity assumptions on the state space).

LEMMA 6.4: *Let c be a bounded, measurable and adapted $\mathbb{R}^d$-valued process defined on some filtered probability space. Then*

i) $\lim_{h\to 0} \mathbb{E}[\int_0^1 |c_t - c_{t-h}|^2 dt] = 0$ *with the convention $c_t = 0$ for $t < 0$.*

ii) There exists a sequence $\delta_n \in [0, 1]$ and a sequence of simple processes c^n such that

$$lim_{n\to\infty} \mathbb{E}[\int_0^1 |c_t - c_t^n|^2 dt] = 0$$

$$\text{and} \quad c_t^n = \sum_{k=0}^{n-1} c_{\frac{k-\delta_n}{n}} \mathbb{I}_{[\frac{k}{n}, \frac{k+1}{n})}(t).$$

Due to the Lipschitz property of V with respect to the wasserstein distance of order p, we have the following approximation result

LEMMA 6.5: *For all $q > 2$ we have $\underset{C\to\infty}{lim}\, z(q, C) = 0$ with*

$$z(q, C) = \sup_{M \in \mathbb{M}^d : |M|=1} \left(r(MM^T) - \sup_{X \sim \nu \in \Delta_0^2 : cov(\nu) = I_d, \|\nu\|_q \leq C} V(MX) \right)$$

Moreover, there exists a measurable selection

$$M \in S_+^d \to \chi(M) \in \{\nu \in \Delta_0^2 : cov(\nu) = I_d\,, \|\nu\|_q \leq C\}$$

such that

$$r(MM^T) - V(M\sharp\chi(M)) \leq |M| z(q, C)$$

where $\nu \to M\sharp\nu$ denotes the map which associates to ν the image probability of ν induced by the linear map $x \to Mx$.

LEMMA 6.6:

Let $(X_k, Y_k)_{k=1,..,n}$ be two $\mathbb{R}^d$-valued martingales defined on the same probability space with respect to the same filtration $(\mathcal{F}_k)_{k=1,..,n}$. Then

$$\frac{1}{\sqrt{n}} \mid \mathcal{V}_n^V(X, \mathcal{F}) - \mathcal{V}_n^V(Y, \mathcal{F}) \mid \leq \beta K \|X_n - Y_n\|_{L^2}$$

where β is a constant such that $|x|_p \leq \beta |x|_2$.

Proof of proposition 6.1. Let B be a d-dimensional Brownian motion and $\overline{\mathcal{F}}^B$ its natural filtration. According to lemma 6.2, for all $\varepsilon > 0$, there exists an $\overline{\mathcal{F}}^B$ martingale $(L_t = \int_0^t \lambda_s dB_s)_{t\in[0,1]}$ such that $[L_1] \preceq \mu$ and an $\overline{\mathcal{F}}^B$-progressively measurable process $(\sigma_t)_{t\in[0,1]}$ such that $\sigma_t\sigma_t^T \in G$ and

$$\mathbb{E}[\langle L_1, \int_0^1 \sigma_s dB_s\rangle] = \mathbb{E}[\int_0^1 Tr(\lambda_s\sigma_s^T)ds] = \mathbb{E}[\int_0^1 r(\lambda_s\lambda_s^T)ds] \geq V_\infty(\mu) - \varepsilon$$

Applying lemma 6.4 to σ, there exists an $\overline{\mathcal{F}}^B$-simple process σ^n such that $\sigma^n(\sigma^n)^T \in G$, piecewise constant on the intervals $[\frac{k}{n}, \frac{k+1}{n})$ such that $\mathbb{E}[\int_0^1 |\sigma_s - \sigma_s^n|^2 ds] \underset{n\to\infty}{\longrightarrow} 0$. Let λ^n be also a piecewise constant approximation of λ by a simple process on the same partition. Let us denote

$$\sigma_s^n = \sum_{k=1}^n c_k^n \mathbb{1}_{[\frac{k-1}{n},\frac{k}{n})}(s) \qquad \lambda_s^n = \sum_{k=1}^n u_k^n \mathbb{1}_{[\frac{k-1}{n},\frac{k}{n})}(s)$$

With these notations and denoting $\Delta_k^n B = B_{k/n} - B_{(k-1)/n}$, we have

$$\mathbb{E}[\langle \int_0^1 \lambda_s^n dB_s, \int_0^1 \sigma_s^n dB_s\rangle] = \mathbb{E}[\sum_{k=1}^n \mathbb{E}[\langle c_k^n \Delta_k^n B, u_k^n \Delta_k^n B\rangle \mid \mathcal{F}^B_{\frac{k-1}{n}}]] \tag{6.3}$$

$$= \mathbb{E}[\sum_{k=1}^n \frac{1}{n} Tr(u_k^n (c_k^n)^T)] \leq \frac{1}{n}\mathbb{E}[\sum_{k=0}^{n-1} r(u_k^n(u_k^n)^T)] \tag{6.4}$$

since $c_k^n(c_k^n)^T \in G$. On the other hand

$$\mathbb{E}[\langle \int_0^1 \lambda_s^n dB_s, \int_0^1 \sigma_s^n dB_s\rangle] \underset{n\to\infty}{\longrightarrow} \mathbb{E}[\langle L_1, \int_0^1 \sigma_s dB_s\rangle]$$

Our goal is to construct a discrete-time approximation of the martingale $\mathbb{E}[L \mid \overline{\mathcal{F}}_t^B]$ using a double-scale of discretization. The first scale is the usual time-discretization on the intervals $[k/n, (k+1)/n)$ and the second level of discretization acts on the integrator B rather than on the integral itself. Each increment $\Delta_k^n B$ will be replaced by a sufficiently long normalized sum of i.i.d. random variables whose laws will be chosen in order for the V-variation to be close to the R'-variation.

Up to enlarging the probability space, we assume that there is a sequence $(V_i)_{i\in\mathbb{N}}$ of uniform random variables independent of B. Let us fix $C > 0$ and $q > 2$. According to lemma 6.3, given a sequence η_n converging to zero, there exists an increasing sequence N_n of integers such that

$$\forall r \geq N_n \sup_{\mu\in RC^r(q,C)} d_{W_2}(\mu, \mathcal{N}(0, I_d)) \leq \eta_n$$

For a vector $(N(k,n))_{k=1,..,n}$ of integers such that $N(k,n) \geq N_n$, define the partial sums $D(k,n) = \sum_{i=1}^k N(i,n)$ and $D(0,n) = 0$. Using the notations of lemma 6.5, define the sequence $(\nu_k^n)_{k=1,..,n}$ of $\mathbb{R}^d$-valued transitions probabilities by $\nu_k^n = \chi(u_k^n)$, having the property that for any variable Y such that $[Y \mid u_k^n] = \nu_k^n$

$$r(u_k^n(u_k^n)^T) - V([u_k^n Y \mid u_k^n]) \leq |u_k^n| z(q, C) \tag{6.5}$$

where $z(q,C)$ is defined in lemma 6.5. There exists a family of random variables $(S_i)_{i=1,..,D(n,n)}$ and a filtration $(\mathcal{H}_i)_{i=1,..,D(n,n)}$ (both depending on n and of the chosen sequence $N(k,n)$)

fulfilling the following properties:

$$\mathbb{E}\left[\left\|\Delta_k^n B - \sum_{i=D(k-1,n)+1}^{D(k,n)} \frac{S_i^n}{\sqrt{nN(k,n)}}\right\|^2 |\mathcal{H}_{D(k-1,n)}\right] \leq \frac{\eta_n^2}{n} \tag{6.6}$$

$$[S_i \mid \mathcal{H}_{i-1}] = \nu_{k^*(i)}^n \in RC^1(q,C) \tag{6.7}$$

$$\mathcal{H}_i = \sigma((u_k^n, \lambda_k^n, \Delta_k^n B), k \leq k^*(i)\,;\, S_j, j \leq i) \text{ for } i = 0,..,D(n,n) \tag{6.8}$$

where $k^*(i)$ is defined by the relation $D(k^*(i)-1,n) \leq i < D(k^*(i),n)$. This construction is simply obtained using the principle of rescaled convolutions explained in lemma 6.3 which implies that there exists a "good" measurable selection

$$\pi(\nu_k^n) \in \mathcal{P}((\nu_k^n)^{\otimes N(k,n)}, \mathcal{N}(0, I_d)).$$

Using the variable V_k (independent from all the past variables), it is then possible to construct the sequence $(S_i)_{i=D(k-1,n)+1,..,D(k,n)}$ such that the conditional law of $((S_i)_{i=D(k-1,n)+1,..,D(k,n)}, \Delta_k^n B)$ given $\mathcal{H}_{D(k-1,n)}$ is $\pi(\nu_k^n)$ [5]. The above mentioned properties follows then from lemma 6.3. Now we can consider the martingale $(M_i = \mathbb{E}[L \mid \mathcal{H}_i], i = 0,..,D(n,n))$ and its approximation

$$\widetilde{M}_i = \sum_{k=1}^{k^*(i)-1} \sum_{j=D(k-1,n)+1}^{D(k,n)} \frac{u_k^n S_j}{\sqrt{nN(k,n)}} + \sum_{j=D(k^*(i)-1,n)+1}^{i} \frac{u_{k^*(i)}^n S_j}{\sqrt{nN(k^*(i),n)}}.$$

which is also an $\mathcal{H}$-martingale. Using lemma 6.6, we have

$$\left|\mathcal{V}_{D(n,n)}^V(M,\mathcal{H}) - \mathcal{V}_{D(n,n)}^V(\widetilde{M},\mathcal{H})\right| \leq \beta K \sqrt{D(n,n)} \left\|L - \widetilde{M}_{D(n,n)}\right\|_{L^2} \tag{6.9}$$

where we replaced $M_{D(n,n)}$ by L using the martingale property. And

$$\begin{aligned}
&\left\|L - \widetilde{M}_{D(n,n)}\right\|_{L^2} \leq \mathbb{E}[\int_0^1 |\lambda_s - \lambda_s^n|^2 ds]^{\frac{1}{2}} \\
&+ \mathbb{E}\left[\sum_{k=1}^n \mathbb{E}[|u_k^n(\Delta_k^n B - \sum_{i=D(k-1,n)+1}^{D(k,n)} \frac{S_i^n}{\sqrt{nN(k+1,n)}})|^2 \mid \mathcal{F}_{k-1/n}^B]\right]^{\frac{1}{2}} \\
&\leq \mathbb{E}[\int_0^1 |\lambda_s - \lambda_s^n|^2 ds] + \alpha\eta_n \mathbb{E}[\int |\lambda_s^n|^2 ds]^{\frac{1}{2}}
\end{aligned}$$

5. See the discussion following theorem 1.3 in the appendix

using (6.6) and where α is such that $|Px| \leq \alpha |P||x|$ for all $P \in \mathbb{M}_d$ and $x \in \mathbb{R}^d$. Using (6.9), these inequalities reduce our problem to the study of the V-variation of $\widetilde{M}$.

$$\begin{aligned}
\mathcal{V}^V_{D(n,n)}(\widetilde{M}, \mathcal{H}) &= \mathbb{E}\left[\sum_{k=1}^{n} \sum_{i=D(k-1,n)+1}^{D(k,n)} V[\widetilde{M}_i - \widetilde{M}_{i-1} \mid \mathcal{H}_{i-1}]\right] \\
&\geq \mathbb{E}\left[\sum_{k=1}^{n} \sum_{i=D(k-1,n)+1}^{D(k,n)} \frac{r(u^n_k (u^n_k)^T) - |u^n_k| z(q,C)}{\sqrt{nN(k,n)}}\right] \\
&\geq \mathbb{E}\left[\sum_{k=1}^{n} \sum_{i=D(k-1,n)+1}^{D(k,n)} \frac{n\mathbb{E}[\langle c^n_k \Delta^n_k B, u^n_k \Delta^n_k B\rangle] - |u^n_k| z(q,C)}{\sqrt{nN(k,n)}}\right] \\
&= \mathbb{E}\left[\sum_{k=1}^{n} \sqrt{nN(k,n)} \left(\mathbb{E}[\langle c^n_k \Delta^n_k B, u^n_k \Delta^n_k B\rangle] - \frac{1}{n}|u^n_k| z(q,C)\right)\right] \\
&\geq \sqrt{D(n,n)} \Big(\mathbb{E}[\int_0^1 Tr(\lambda^n_s (\sigma^n_s)^T) ds] - z(q,C)\mathbb{E}[\int_0^1 |\lambda^n_s| ds] \\
&\quad - \max_{k=1,..,n} \left|\frac{\sqrt{nN(k,n)}}{\sqrt{D(n,n)}} - 1\right| \left|\mathbb{E}[\int_0^1 Tr(\lambda^n_s (\sigma^n_s)^T) ds] - z(q,C)\mathbb{E}[\int_0^1 |\lambda^n_s| ds]\right|\Big)
\end{aligned}$$

where the two first inequalities follow from (6.5) and (6.3). Using the former results, for any sequence of vectors $N(k,n)$ indexed by n such that

$$\max_{k=1,..,n} \left|\frac{\sqrt{nN(k,n)}}{\sqrt{D(n,n)}} - 1\right| \underset{n\to\infty}{\longrightarrow} 0 \tag{6.10}$$

we have

$$\liminf_{n\to\infty} \frac{\mathcal{V}^V_{D(n,n)}(\widetilde{M}, \mathcal{H})}{\sqrt{D(n,n)}} \geq V_\infty(\mu) - z(q,C)\mathbb{E}[\int_0^1 |\lambda_s| ds] - \varepsilon$$

The condition (6.10) is not restrictive since for fixed n, any vector of integer $N(k,n) \in \{m; m+1\}$ for $m \geq N_n$ is such that

$$\max_{k=1,..,n} \left|\frac{\sqrt{nN(k,n)}}{\sqrt{D(n,n)}} - 1\right| \leq \frac{2}{N_n}$$

and then any value above nN_n is admissible for $D(n,n)$. It implies that

$$\liminf_{n\to\infty} V_n(\mu) \geq V_\infty(\mu) - z(q,C)\mathbb{E}[\int_0^1 |\lambda_s| ds] - \varepsilon$$

The result follows by sending C to $+\infty$ and ε to 0. □

7. The dual problem : Dual equality

This section is devoted to prove the following dual representation of V_∞

PROPOSITION 7.1:

$$V_\infty(\mu) = \underset{\phi\in C_2(\mathbb{R}^d)}{inf} (\mathbb{E}_\mu[\phi(L)] + V^*_\infty(\phi)) \tag{7.1}$$

$$= \underset{\phi\in Conv(\mathbb{R}^d)}{min} (\mathbb{E}_\mu[\phi(L)] + V^*_\infty(\phi)) \tag{7.2}$$

where

$$V^*_\infty(\phi) \triangleq \underset{[(Z_t)_{t\in[0,1]}]\in Q_\Gamma}{sup} \mathbb{E}[\phi^*(Z_1)]$$

$C_2(\mathbb{R}^d)$ is the set of continuous functions with at most quadratic growth and $Conv(\mathbb{R}^d)$ the set of l.s.c. proper convex functions from $\mathbb{R}^d$ to $\mathbb{R}\cup\{+\infty\}$.

NOTATION 8: *Let $\partial V_\infty(\mu)$ denote the set of $\phi \in Conv(\mathbb{R}^d)$ minimizing the right-hand side of (7.1).*

PROOF. Note at first that we can replace the constraint $[L] \preceq \mu$ given in the definition of V_∞ by $[L] = \mu$ since the maximal covariance functions $C(.,\nu)$ defined in section 12 are Blackwell increasing (see lemma 12.4). Moreover, it's sufficient to prove the result for $\mu \in Delta^2_0$ since the set $Q_\Gamma(1)$ contains only centered probabilities. Applying then theorem 12.1 using the notation $\langle\phi,\mu\rangle = \mathbb{E}_\mu[\phi(L)]$ for expectations, we obtain

$$V_\infty(\mu) = \underset{\nu\in Q_\Gamma(1)}{max} \quad \underset{(\phi-\frac{1}{2}|.|^2,\psi-\frac{1}{2}|.|^2)\in C_b(\mathbb{R}^2)^2;\phi+\psi\geqslant\langle.,.\rangle}{inf} (\langle\phi,\mu\rangle + \langle\psi,\nu\rangle)$$

$Q_\Gamma(1)$ is a compact subset of $\Delta^2(\mathbb{R}^d)$, and weak convergence coincides in this set with the d_{W_2}-convergence since since moments of order $q > 2$ are uniformly bounded. Therefore and since the function ψ in the above expression of $V_\infty(\mu)$ has at most quadratic growth, the application

$$\nu \to \mathbb{E}_\nu[\psi(Z_1)]$$

is affine and continuous on $Q_\Gamma(1)$. On the other hand, the application

$$(\phi,\psi) \to (\langle\phi,\mu\rangle + \langle\psi,\nu\rangle)$$

is affine on the convex set $\{(\phi,\psi) \in (\frac{1}{2}|.|^2 + C_b(\mathbb{R}^d)) \times (\frac{1}{2}|.|^2 + C_b(\mathbb{R}^d)) : \phi+\psi \geqslant \langle.,.\rangle\}$, so that the minmax theorem ([**53**]) implies:

$$V_\infty(\mu) = \underset{(\phi-\frac{1}{2}|.|^2,\psi-\frac{1}{2}|.|^2)\in C_b(\mathbb{R}^d)^2;\phi+\psi\geqslant\langle.,.\rangle}{inf} (\langle\phi,\mu\rangle + \underset{\nu\in Q_\Gamma(1)}{max}\langle\psi,\nu\rangle)$$

For a function f defined on $\mathbb{R}^d$, let f^* denote the Fenchel transform of f. Since for any pair (ϕ,ψ) we have $(\phi^*)^* \le \phi$ and $\phi^* \le \psi$, we infer:

$$V_\infty(\mu) = \underset{\phi}{inf}(\langle\phi,\mu\rangle + \underset{\nu\in Q_\Gamma(1)}{sup}\langle\phi^*,\nu\rangle) \tag{7.3}$$

where the infimum is taken over convex functions $\phi \in \frac{1}{2}|.|^2 + C_b(\mathbb{R}^d)$. Finally, equality still holds for $\phi \in C_2(\mathbb{R}^d)$ or $Conv(\mathbb{R}^d)$ using Fenchel's lemma.

Let now (ϕ_n,ψ_n) be a minimizing sequence with $\phi_n \in \frac{1}{2}|.|^2 + C_b(\mathbb{R}^d)$ and $\psi_n = \phi^*_n$. Replacing (ϕ_n,ψ_n) by $(\phi_n - \phi_n(0), \psi_n + \phi_n(0))$, we can assume that

$$\forall y \in \mathbb{R}^d, \quad \psi_n(y) \ge 0 \quad \phi_n(0) = 0$$

The two sequences $\langle\phi_n,\mu\rangle$ and $\underset{\nu\in Q_\Gamma(1)}{sup}\langle\psi_n,\nu\rangle$ are therefore bounded from below since μ is centered and

$$\forall x\in\mathbb{R}^d,\ \phi_n(x)\geq\langle u_n,x\rangle,\ \text{with } u_n\in\partial\phi_n(0).$$

Since their sum converges to $V_\infty(\mu)$, they are also bounded from above. Precisely, we have

$$\underset{n\in\mathbb{N}}{sup}\ \underset{\nu\in Q_\Gamma(1)}{sup}\langle\psi_n,\nu\rangle<\infty$$

All the probabilities in $Q_\Gamma(1)$ being centered, it follows from Jensen's inequality that

$$\beta=\underset{n\in\mathbb{N}}{sup}\,\psi_n(0)<\infty.$$

Let B_r be the ball of center 0 and radius r in $\mathbb{R}^d$. We prove next that for all $r>0$, we have

$$\underset{n\in\mathbb{N}}{sup}\ \underset{y\in B_r,\,x\in\partial\psi_n(y)}{sup}|x|<\infty$$

Suppose on the contrary that there exists a sequence (y_k,x_k) such that $y_k\in B_r$, $x_k\in\partial\psi_{n(k)}(y_k)$ for some sequence of integers $n(k)$ and $|x_k|\to\infty$. This implies

$$\psi_{n(k)}(y)\geq\psi_{n(k)}(y_k)+\langle x_k,y-y_k\rangle\geq\langle x_k,y-y_k\rangle$$

On the line $\{y=t\frac{x_k}{|x_k|}, t\in\mathbb{R}\}$, we have

$$\psi_{n(k)}(y)\geq\begin{cases}0 & \text{if } t\leq r\\ \langle x_k,y-\frac{rx_k}{|x_k|}\rangle & \text{if } t>r\end{cases}$$

Using that Γ is a neighborhood of 0, there exists ν_k in $Q_\Gamma(1)$ which is a normal distribution on the line $\{y=t\frac{x_k}{|x_k|}, t\in\mathbb{R}\}$ with variance greater than some constant $\varepsilon>0$ independent of k. To prove this, note that for any vector x, the law of a Brownian motion with covariance λxx^T is in Q_Γ for some sufficiently small $\lambda>0$ and that the variance can be bounded from below independently of the direction of x using that Γ is a neighborhood of 0. We deduce that

$$\langle\psi_{n(k)},\nu_k\rangle\to\infty$$

using a change of variable formula and monotone convergence, which brings a contradiction. Ascoli's theorem implies therefore that the sequence ψ_n is relatively compact in $C(\mathbb{R}^d)$ for the uniform convergence on compact sets. Let ψ denote the limit of some convergent subsequence also denoted ψ_n. Pointwise convergence implies that $\psi(y)\geq 0$ for all $y\in\mathbb{R}^d$, and we deduce therefore from Fatou's lemma that

$$\underset{\nu\in Q_\Gamma(1)}{sup}\langle\psi,\nu\rangle\leq\underset{n\to\infty}{liminf}\ \underset{\nu\in Q_\Gamma(1)}{sup}\langle\psi_n,\nu\rangle$$

For $l\in\mathbb{N}$, let ξ_{B_l} the convex indicator function equal to 0 on B_l and $+\infty$ otherwise. For any function f, we define $f^{*l}=(f+\xi_{B_l})^*$, so that the sequence f^{*l} is nondecreasing and converges pointwise to f^*. Using that Fenchel transform is an isometry for the uniform norm, l being fixed, ψ_n^{*l} converges uniformly to ψ^{*l} when n goes to $+\infty$. Using these notations

$$\int\psi^{*l}d\mu=\underset{n\to\infty}{lim}\int\psi_n^{*l}d\mu\leq\underset{n\to\infty}{liminf}\int\psi_n^*d\mu$$

Monotone convergence implies $\underset{l\to\infty}{lim} \int \psi^{*l} d\mu = \int \psi^* d\mu$, then:

$$\int \psi^* d\mu \leq \underset{n\to\infty}{liminf} \int \phi_n d\mu$$

Finally, the pair (ψ^*, ψ) is optimal. $\square$

The next result is quite similar to the characterization given in theorem 12.2.

LEMMA 7.1: *In the following, $\phi \in Conv(\mathbb{R}^d)$, μ denotes the law of the variable L in Δ^2, and Z is a process whose law is in Q_Γ, both defined on the same probability space. The two following assertions are equivalent*

$$i) L \in \partial\phi^*(Z_1) \text{ almost surely, and } \mathbb{E}[\phi^*(Z_1)] = \underset{\nu\in Q_\Gamma(1)}{sup} \langle \phi^*, \nu\rangle$$

ii) The joint distribution of (L, Z_1) is optimal for $V_\infty(\mu)$ and $\phi \in \partial V_\infty(\mu)$.

PROOF. It follows directly from the definition of V_∞ and Fenchel's lemma. Indeed, suppose ii)

$$V_\infty(\mu) = \mathbb{E}[\langle L, Z_1\rangle] \leq \mathbb{E}[\phi(L) + \phi^*(Z_1)] \leq \langle \phi, \mu\rangle + \underset{\nu\in Q_\Gamma(1)}{sup} \langle \phi^*, \nu\rangle = V_\infty(\mu)$$

Therefore, all the above inequalities are equalities, and $\langle L, Z_1\rangle = \phi(L)+\phi^*(Z_1)$ with probability 1 which proves the result by Fenchel's lemma. Conversely, if i) is true, then it follows from (7.1) that

$$V_\infty(\mu) \geq \mathbb{E}[\langle L, Z_1\rangle] = \mathbb{E}[\phi(L) + \phi^*(Z_1)] = \langle \phi, \mu\rangle + \underset{\nu\in Q_\Gamma(1)}{sup} \langle \phi^*, \nu\rangle \geq V_\infty(\mu)$$

which ends the proof. $\square$

8. The dual problem : PDE formulation.

The aim of this section is to characterize V_∞^* as a second-order nonlinear PDE problem (HJB) and to state a comparison result associated with this HJB equation.

We know from proposition 7.1 and lemma 7.1 that all the optimizers of V_∞ are linked with the optimizers of the dual problem V_∞^*. Moreover, the set of dual variables ψ can be restricted to the set of functions such that $\psi^* = f$ is a real-valued convex function defined on $\mathbb{R}^d$ and

$$\forall x, \quad f(x) \geq -\frac{|x|^2}{2}; \qquad \underset{\mathbb{P}\in Q_\Gamma}{sup} \mathbb{E}_\mathbb{P}[f(X_1)] < \infty$$

Define the associated time-dependent value function

$$u : (0,1] \times \mathbb{R}^d \longrightarrow \mathbb{R} : (t,x) \longrightarrow \underset{\mathbb{P}\in Q_\Gamma}{sup} \mathbb{E}_\mathbb{P}[f(x + X_{1-t})]$$

Then we have, using ∂_t for the time derivative and ∇^2 for the spatial hessian matrix :

PROPOSITION 8.1: *The function u is the unique continuous viscosity solution of*

$$\begin{cases} -\partial_t u - \sup_{P\in\Gamma} Tr(P\nabla^2 u) & = & 0 & in & (0,1)\times\mathbb{R}^d \\ u(1,x) & = & f(x) & in & \mathbb{R}^d \end{cases} \tag{8.1}$$

in the class of functions $\mathfrak{C}$ with the following restriction on growth:

$$u \in \mathfrak{C} \Longleftrightarrow \forall t\in(0,1),\ \exists M,\rho>0,\ \forall(s,x)\in[t,1]\times\mathbb{R}^d,\ |u(s,x)|\le Me^{\rho|x|^2}$$

Moreover, $V^*_\infty(f^*) = \sup_{\mathbb{P}\in Q_\Gamma}\mathbb{E}_{\mathbb{P}}[f(X_1)] = \lim_{t\to 0} u(t,0)$.

The proof follows from the usual method: at first we prove that u satisfies a dynamic programming principle (DPP), and we deduce from this that u is a viscosity solution of (8.1). Then we show that equation (8.1) satisfies a comparison principle. All of these proofs follow the usual methods of stochastic control, and this equation appears in many recent works in relation with the G-expectation (see [**47**] and [**33**]), but our particular case is not explicitly treated due to our particular growth condition on the boundary condition f. Therefore we give proof of these results for the sake of completeness. Moreover the following lemma which gives a characterization of the set Q_Γ in terms of solutions of a a submartingale problem allows to give a short and simple proof for the DPP. Let us define the support function of Γ in S^d_+ by

$$S_\Gamma : S^d \longrightarrow \mathbb{R}_+ \ :\ Q \longrightarrow \sup_{P\in\Gamma} Tr(PQ)$$

Note that S_Γ is nonnegative since $0\in\Gamma$.

LEMMA 8.1: *For a probability $\mathbb{P}\in\mathcal{M}^c$, we have*

$$\mathbb{P}\in Q_\Gamma \Longleftrightarrow \forall\phi\in C^\infty_c(\mathbb{R}^d),\ (Y^\phi_t)_{t\in[0,1]} \text{ is a submartingale.} \tag{8.2}$$

where $C^\infty_c(\mathbb{R}^d)$ is the set of real-valued infinitely differentiable functions with compact support and

$$Y^\phi_t = \int_0^t S_\Gamma(\nabla^2\phi(X_s))ds - 2\phi(X_t)$$

PROOF. If $\mathbb{P}\in Q_\Gamma$, the first implication follows easily form Ito's formula applied to $\phi(X_t)$ and using remark 5.1. For the converse implication, note that the property extends to the set $C^2_b(\mathbb{R}^d)$ of twice continuously differentiable functions with bounded derivatives using Lebesgue's dominated convergence theorem. Then given $M\in S^d_+$, there exists a sequence $\phi_n\in C^2_b(\mathbb{R}^d)$ such that

$$\forall x\in\mathbb{R}^d, \qquad 0\le\nabla^2\phi_n(x)\le M$$

and $\nabla^2\phi_n$ is nondecreasing and converges to M. Y^{ϕ_n} is a submartingale bounded in L^2 for each n and therefore admits a unique Doob-Meyer decomposition as the sum of a martingale and a nondecreasing predictable process. This nondecreasing process is equal from Ito formula to

$$V^{\phi_n}_t = \int_0^t S_\Gamma(\nabla^2\phi_n(X_s))ds - Tr(\int_0^t \nabla^2\phi_n(X_s)d\langle X\rangle_s)$$

Therefore for $s<t$ we have

$$\int_s^t S_\Gamma(\nabla^2\phi_n(X_u))du - Tr(\int_s^t \nabla^2\phi_n(X_u)d\langle X\rangle_u) \ge 0$$

And, using monotone convergence for almost all ω, we obtain when n goes to ∞:

$$(t-s)\left(S_\Gamma(M) - Tr(M(\frac{\langle X\rangle_t - \langle X\rangle_s}{t-s}))\right) \geq 0$$

Finally, using the continuity of $\langle X\rangle$ and S_Γ, this property holds with probability 1 for all rational numbers $s,t \in [0,1]$ and M in a countable dense subset of S_+^d, which proves (5.1) since Γ is closed and convex. ☐

We deduce from this characterization useful stability properties of Q_Γ. (Recall definition 5.1 for the notations.)

LEMMA 8.2: *Let $s \in [0,1]$ and Z a martingale defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, \mathbb{P})$ whose law is in Q_Γ. Then the conditional law of $(Z_t - Z_s)_{t\geq s}$ given any countably-generated sub-σ-field of $\mathcal{F}_s$ is almost surely in $\pi_{1-s}(Q_\Gamma)$. On the other hand, if the process $(Y_u, u \in [0,1-s])$ is such that its conditional distribution given $\sigma(Z_u, u \leq s)$ has values in Q_Γ with probability 1, the law of the process $\widehat{Z}_s = Z_{s\wedge t} + Y_{(s\vee t)-t}$ is in Q_Γ.*

PROOF. The first assertion follows directly from theorem 1.2.10 in [**56**] applied to Y^ϕ with ϕ in a countable dense subset of $C_c^\infty(\mathbb{R}^d)$. The second one follows from lemma 6.1.1 in the same reference or lemma III.2.47 in [**36**] with slight modifications. ☐

Now we proceed to the Dynamic Programming Principle

LEMMA 8.3: *Let $t \in (0,1)$ and τ be a $[0,1-t]$-valued stopping time of the (non right-continuous) filtration $\mathcal{F}_t = \sigma(Z_s, s \leq t)$. Then*

$$u(t,x) = \underset{\mathbb{P}\in Q_\Gamma}{sup}\,\mathbb{E}_\mathbb{P}[u(t+\tau, x+Z_\tau)]$$

Moreover, u is real-valued and l.s.c on $(0,1]\times\mathbb{R}^d$ and belongs to $\mathfrak{C}$.

PROOF. Let us modify the definition of Q_Γ for this proof, assuming to shorten notations that it is a subset of $\Delta(C([0,\infty),\mathbb{R}^d))$ and that (5.1) is valid on the whole real line. Define the function

$$J(t,x,\mathbb{P}) : [0,1]\times\mathbb{R}^d\times Q_\Gamma \longrightarrow \mathbb{R}\cup\{+\infty\} : (t,x,\mathbb{P}) \longrightarrow \mathbb{E}_\mathbb{P}[f(x+Z_{1-t})]$$

At first, J is well-defined since $f(x) \geq -\frac{|x|^2}{2}$ and l.s.c. in $(t,x,\mathbb{P})$ using Fatou's lemma. Moreover, J is convex in x since f is convex, and nonincreasing in t using Jensen's inequality and the martingale property of Z. We deduce that the function u (extended in $t=0$) shares the same properties since

$$u(t,x) = \underset{\mathbb{P}\in Q_\Gamma}{sup}\ J(t,x,\mathbb{P})$$

Using now the stability of properties of Q_Γ given in lemma 8.2, we have that for any τ as defined above

$$J(t,x,\mathbb{P}) = \mathbb{E}_\mathbb{P}[\mathbb{E}_\mathbb{P}[f(x+Z_{1-t}) \mid \mathcal{F}_\tau]] \leq \mathbb{E}_\mathbb{P}[u(x+X_\tau, t+\tau)]$$

which proves the first part of the inequality. For the second, using theorem 2.2 in the appendix, there exists for all $\varepsilon > 0$ an ε-optimal measurable selection $H^\varepsilon(t,x) \in Q_\Gamma$, i.e. such that

$$\forall (t,x) \in [0,1] \times \mathbb{R}^d, \quad J(t,x,H^\varepsilon(t,x)) \geq \begin{cases} u(t,x) - \varepsilon & if \quad u(t,x) < +\infty \\ \frac{1}{\varepsilon} & if \quad u(t,x) = +\infty \end{cases}$$

This allows to construct a probability $\mathbb{P}_\varepsilon$ that coincides with $\mathbb{P}$ on $\mathcal{F}_\tau$ using the above properties such that

$$\mathbb{E}_{\mathbb{P}_\varepsilon}[f(x + Z_{1-t}) \mid \mathcal{F}_\tau] \geq \begin{cases} u(x + Z_\tau, t+\tau) - \varepsilon & if \quad u(x + Z_\tau, t+\tau) < +\infty \\ \frac{1}{\varepsilon} & if \quad u(x + Z_\tau, t+\tau) = +\infty \end{cases}$$

and this proves the result by sending ε to zero. Using the condition $u(0,0) < +\infty$, we will prove that the function u is real-valued on $(0,1] \times \mathbb{R}^d$. Suppose on the contrary that $u(t_0, x_0) = +\infty$. Since u is convex in x, $u(t_0, x)$ is infinite on an affine half-space H. Let $M \in \Gamma \cap S^d_{++}$ (recall that Γ is a neighborhood of 0), and $\mathbb{P}_M \in Q_\Gamma$ be a Brownian motion with constant covariance M. This implies that the law of Z_t has a full support in $\mathbb{R}^d$. Construct $\mathbb{P}_\varepsilon$ as above with $\tau = t$. We have

$$\mathbb{E}_{\mathbb{P}_\varepsilon}[f(Z_1) \mid \mathcal{F}_t] \geq \begin{cases} u(Z_t, t) - \varepsilon & if \quad u(Z_t, t) < +\infty \\ \frac{1}{\varepsilon} & if \quad u(Z_t, t) = +\infty \end{cases}$$

But $\mathbb{P}_\varepsilon(Z_t \in H) = \mathbb{P}_M(Z_t \in H) > 0$ and this implies

$$\mathbb{E}_{\mathbb{P}_\varepsilon}[f(Z_1)] \underset{\varepsilon \to 0}{\longrightarrow} +\infty$$

which contradicts our hypothesis. That u belongs to $\mathfrak{C}$ follows then from the fact that for $t \in (0,1)$, we have $\mathbb{E}_{\mathbb{P}_M}[u(s, Z_t)] < \infty$ for all $t < 1 - s$. □

In order to prove that the boundary condition is satisfied by u, we need the following domination result.

LEMMA 8.4: *There exists $P \in S^d_{++}$ such that $\forall M \in \Gamma$, $M \leq P$. Let $\mathbb{P}_P$ be the law of a Brownian motion $(B_t)_{t \in [0,1]}$ with covariance P, then for all processes Z with law $\nu \in Q_\Gamma$, we have $[Z_t] \preceq [B_t]$ for all $t \in [0,1]$. Moreover, for sufficiently small $\varepsilon > 0$, the law of $(B_{t\varepsilon})_{t \in [01]}$ is in Q_Γ.*

PROOF. Consider a process Z with a law in Q_Γ. Let us enlarge the probability space and consider a d-dimensional standard Brownian motion W independent of Z. From the definition of Q_Γ, there exists an $\mathcal{F}^Z$-progressively measurable process ρ with values in Γ such that

$$\langle Z \rangle_t = \int_0^t \rho_s ds$$

Define then

$$Y_t = \int_0^t \sqrt{P - \rho_s} dW_s$$

The quadratic covariation process $\langle Z, Y \rangle$ satisfies $\langle Z, Y \rangle = 0$ by construction and $Z + Y$ follows the law $\mathbb{P}_P$. Now it remains to prove that $\mathbb{E}[Y_1 \mid Z_1] = 0$. But this follows from the fact that the conditional law of Y_1 given $\mathcal{F}^W_1$ is a centered gaussian distribution [6]. For the second assertion,

6. One can prove this result directly by approximations with integrals of simple processes, or deduce it from proposition 1.1 in [**35**] which is much more general.

using the scaling property of the Brownian motion, the process $B_{t\varepsilon}$ is a Brownian motion with covariance εP, which belongs to Γ for small ε. $\square$

LEMMA 8.5: *For all $x \in \mathbb{R}^d$, $u(t,y) \underset{t\to 1,\, y\to x}{\longrightarrow} f(x)$.*

PROOF. Since Q_Γ is compact and u is l.s.c., we only have to prove that

$$\forall (t_n, x_n, \mathbb{P}_n) \underset{n\to\infty}{\longrightarrow} (1, x, \mathbb{P}), \quad J(t_n, x_n, \mathbb{P}_n) \underset{n\to\infty}{\longrightarrow} f(x)$$

Let $h(x) = \frac{1}{2}|x^2|$. Using lemma 1.3 in the appendix, the function $(t, x, \mathbb{P}) \to \mathbb{E}_\mathbb{P}[h(x + Z_{1-t})]$ is continuous, and our problem reduces then to prove that

$$\mathbb{E}_{\mathbb{P}_n}[\tilde{f}(x_n + Z_{1-t_n})] \underset{n\to\infty}{\longrightarrow} \mathbb{E}_\mathbb{P}[\tilde{f}(x + Z_0)] = \tilde{f}(x)$$

where $\tilde{f} = f + g$ is by construction a nonnegative convex function. To prove this, it is sufficient to prove that the sequence is uniformly integrable, i.e.

$$\underset{M\to\infty}{lim} \ \underset{n\in\mathbb{N}}{sup} \ \mathbb{E}_{\mathbb{P}_n}[(\tilde{f}(x_n + Z_{1-t_n}) - M)^+] = 0$$

Let $\varepsilon > 0$ small and P be given by lemma 8.4, and n sufficiently large so that $1 - t_n \le \varepsilon/2$. Then since $(\tilde{f} - M)^+$ is convex, lemma 8.4 implies

$$\mathbb{E}_{\mathbb{P}_n}[(\tilde{f}(x_n + Z_{1-t_n}) - M)^+] \le \mathbb{E}_{\mathbb{P}_P}[(\tilde{f}(x_n + B_{\varepsilon/2}) - M)^+]$$

Since $\tilde{f}$ is convex, and using a finite sequence $(y_i \in \mathbb{R}_d)_{i=1,..,I}$ such that

$$\{x_n\}_{n\in\mathbb{N}} \subset co(\{y_i,\ i \in I\})$$

we have

$$\mathbb{E}_{\mathbb{P}_P}[(\tilde{f}(x_n + B_{\varepsilon/2}) - M)^+] \le \sum_{i=1}^{I} \mathbb{E}_{\mathbb{P}_P}[(\tilde{f}(y_i + B_{\varepsilon/2}) - M)^+]$$

and the right-hand side converges to 0 when M goes to ∞ since for all y

$$\mathbb{E}_{\mathbb{P}_P}[f(y + B_{\varepsilon/2})] \le u(1/2, y) < \infty$$

$\square$

The DPP implies that u is a viscosity solution of the corresponding equation. We give a proof below of this standard result.

LEMMA 8.6: *The function u is a (discontinuous) viscosity solution of* (8.1)

PROOF. At first, the boundary condition is satisfied using the previous lemma. Next, if u is not a supersolution of (8.1), then there exists a smooth function ϕ and $(t,x) \in (0,1) \times \mathbb{R}^d$ such that $min(u - \phi) = 0$ is reached in (t,x) and $-\partial\phi(t,x) - S_\Gamma(\nabla^2\phi(t,x)) < 0$. There exists therefore $M \in \Gamma$ such that $-\partial\phi(t,x) - Tr(M\nabla^2\phi(t,x)) < 0$. This inequality is still true on a bounded open neighborhood $[t, t+h) \times B(x,r)$ of (t,x). Let $\mathbb{P}_M \in Q_\Gamma$ be the law of a Brownian motion with covariance M and $\theta = min(h, T_{B(0,r)^c})$ with $T_{B(0,r)^c}$ the hitting time of $B(0,r)^c$ for the canonical process Z. Ito formula implies (the martingale term being integrable because of the definition of θ)

$$\mathbb{E}_\mathbb{P}[\phi(t+\theta, x + Z_\theta)] = \phi(t,x) + \mathbb{E}_\mathbb{P}[\int_0^\theta \left(\partial_t\phi(t+s, x+Z_s) + Tr(M\nabla^2\phi(t+s, x+Z_s))\right) ds]$$

Since $u \geq \phi$ and the last term is positive, this contradicts the DPP.
If the s.c.s. envelope u^* of u is not a sub-solution of (8.1), then there exists a smooth function ϕ and $(t,x) \in (0,1) \times \mathbb{R}^d$ such that $max(u-\phi) = 0$ is reached in (t,x) and $-\partial\phi(t,x) - S_\Gamma(\nabla^2\phi(t,x)) \geq \eta > 0$. This inequality is still true on a bounded open neighborhood $(t-h, t+h) \times B(x,r)$ of (t,x) (up to changing the definition of η). Now, let (t_n, x_n) a sequence converging to (t,x) and such that $u(t_n,x_n)$ converge to $u^*(t,x)$. We can assume that $(t_n,x_n) \in (t-h,t+h) \times B(x,r)$ for all n. Define $\theta_n = min(t+h-t_n, T_{B(0,r)^c})$, then for all $\mathbb{P} \in Q_\Gamma$, Ito formula implies

$$\begin{aligned}\phi(t_n,x_n) \geq & \mathbb{E}_\mathbb{P}[\phi(t_n+\theta_n, x_n+Z_{\theta_n})] \\ & - \mathbb{E}_\mathbb{P}[\int_0^{\theta_n} \Big(\partial_t\phi(t_n+s,x_n+Z_s) + S_\Gamma(\nabla^2\phi(t_n+s,x_n+Z_s))\Big)\,ds] \\ \geq & \eta + \mathbb{E}_\mathbb{P}[u(t_n+\theta_n, x_n+Z_{\theta_n})]\end{aligned}$$

Since $\phi(t_n,x_n)$ and $u(t_n,x_n)$ converge to the same limit $u^*(t,x)$, for n sufficiently large we have

$$u(t_n,x_n) \geq \frac{\eta}{2} + \mathbb{E}_\mathbb{P}[u(t_n+\theta_n, x_n+Z_{\theta_n})]$$

which contradicts the DPP since we can take the supremum over $\mathbb{P}$ in the right-hand side. □

And, finally, the following lemma [7] states a comparison result using the classical solutions of the heat equation.

Lemma 8.7: *Equation* (8.1) *admits a unique continuous viscosity solution in the class* $\mathfrak{C}$.

Proof. It is clearly sufficient to prove uniqueness in $[T,1) \times \mathbb{R}^d$ for all $T \in (0,1)$ in the class of locally bounded functions $C(M,\rho)$ satisfying $|u(t,x)| \leq Me^{\rho|x|^2}$ for some $M, \rho > 0$. At first, up to replace u by $w(x,t) = u(\alpha x, t)$ for some sufficiently large α, we can assume that $|P| \leq 1$ for $P \in \Gamma$, and in particular $Tr(P) \leq d$. Suppose that v and w are respectively u.s.c. sub-solution and l.s.c super-solutions of (8.1) in $C(M,\rho)$, we will prove that $v \leq w$ which as usual implies uniqueness. At first, we have that $z = v - w$ is a sub-solution of the equation

$$\begin{cases} -\partial_t u - \sup\limits_{P\in\Gamma} Tr(P\nabla^2 v) = 0 & \text{in} \quad [T,1) \times \mathbb{R}^d \\ \qquad u(1,x) = 0 & \text{in} \qquad \mathbb{R}^d \end{cases} \tag{8.3}$$

For a proof, we refer to lemma 1.2 in [**21**] which is far more general (all the technical difficulties are actually contained in this result, whose proof relies on Ishii's lemma, and the main assumption we need to apply it is that Γ is bounded). The second ingredient is the following classical solution (see [**37**] p.217).

Claim 8.1: *For* $\mu, \varepsilon > 0$ *the function*

$$\phi_\mu(t,x) = \frac{\mu}{(t-T+\varepsilon)^{d/2}} \exp\left(\frac{|x|^2}{4(t-T+\varepsilon)}\right)$$

is a smooth supersolution of (8.3).

7. This proof was communicated to us by Olivier Ley (of course, any mistake is ours). The result seems to be well-known but we were not able to find a reference covering exactly this case. Let us mention however that this results is implicitly contained in [**6**] (see remark 3.6) in which a considerably wider class of equations is considered.

Proof of claim 8.1. Terminal condition is met since $\phi_\mu \geq 0$, and

$$-\partial_t\phi_\mu(t,x) - S_\Gamma(\nabla^2\phi_\mu(t,x)) = \frac{\mu}{(t-T+\varepsilon)^{d/2}} e^{\left(\frac{|x|^2}{4(t-T+\varepsilon)}\right)} \left(\frac{d - S_\Gamma(I_d)}{2(t-T+\varepsilon)} + \frac{|x|^2 - S_\Gamma(xx^T)}{4(t-T+\varepsilon)^2}\right)$$

By hypothesis $\underset{P\in\Gamma}{sup}\, Tr(P) = S_\Gamma(I_d) \leq d$ and by Cauchy-Schwartz

$$S_\Gamma(xx^T) \leq 1\sqrt{Tr(xx^T)^2} \leq |x|^2$$

which proves the result. $\square$

Suppose that T is sufficiently close to 1 so that $\rho < \frac{1}{4(1-T)}$ and for small enough $\varepsilon > 0$, we have $\rho < \frac{1}{4(1-T+\varepsilon)}$. This implies

$$z(t,x) - \phi_\mu(t,x) \leq 2Me^{\rho|x|^2} - \frac{\mu}{(1-T+\varepsilon)^{N/2}} e^{\frac{|x|^2}{4(1-T+\varepsilon)}} \underset{|x|\to\infty}{\longrightarrow} -\infty$$

Therefore, for all $\eta > 0$, there exists $(\bar{t}, \bar{x}) \in [T,1]\times\mathbb{R}^d$ such that

$$\underset{[T,1]\times\mathbb{R}^d}{sup} (z(t,x) - \phi_\mu(t,x) - \eta(1-t)) = (z(\bar{t},\bar{x}) - \phi_\mu(\bar{t},\bar{x}) - \eta(1-\bar{t}))$$

If $\bar{t} < 1$, the definition of z being a sub solution of (8.3) applied to the test function $\phi_\mu + \eta(1-t)$ gives in $(\bar{t},\bar{x})$

$$-\partial_t\phi_\mu + \eta - S_\Gamma(\nabla^2\phi_\mu) \leq 0$$

which is absurd from lemma 8.1 and since $\eta > 0$. We conclude that $\bar{t} = 1$, which means

$$\forall (t,x) \in [T,1)\times\mathbb{R}^d, \qquad z(t,x) - \phi_\mu(t,x) - \eta(1-t) \leq z(1,\bar{x}) - \phi_\mu(1,\bar{x}) \leq 0$$

This implies $z(t,x) \leq \phi_\mu(t,x) + \eta(1-t)$ and then $z(t,x) \leq 0$ by sending η,μ to zero, which was the desired inequality. To conclude, this proof works only for T close to 1, but in the general case, if T' is sufficiently close to 1, the above proof shows uniqueness on $[T',1]$ and since $z(T',x) \leq 0$, the argument can be repeated on the interval $[T'-(1-T'),T']$, and this allows to conclude by induction. Finally, this result also proves that u is continuous. Indeed, using the former notations, the function $u(t,x)$ is l.s.c. and thus a super-solution of (8.1) and its u.s.c. envelope $u^*(t,x)$ is a sub-solution (lemma 8.5 is needed here to prove that u^* fulfills the boundary condition). Since by definition $u^* \geq u$, we conclude that $u = u^*$ and thus u is continuous. $\square$

9. Asymptotic distributions : A first reduction

Before going to the proof of theorem 1.2, we need to recall some useful properties properties of the Meyer-Zheng topology (see [**45**]) on the space of martingale distributions. This topology is defined on the set $\mathbb{D}([0,1],\mathbb{R}^d)$ of cádlág functions as the convergence in measure with respect to Lebesgue's measure (denoted λ) together convergence of the value at time 1 : a sequence y_n converges to y if

$$\forall\varepsilon > 0,\ \lambda(\{|y_n(x) - y(x)| \geq \varepsilon\}) \underset{n\to\infty}{\longrightarrow} 0, \qquad \text{and} \quad y_n(1) \underset{n\to\infty}{\longrightarrow} y(1)$$

The following lemma summarizes some useful properties of the M-Z topology.

LEMMA 9.1: *The sets of martingale's distributions uniformly bounded in L^q for some $q > 1$ are compact in $\Delta(\mathbb{D}([0,1],\mathbb{R}^d))$ when endowed with the topology of convergence in distribution associated to the M-Z topology (that will be also called M-Z for simplicity). This implies as a corollary that the set $\mathcal{M}(\preceq_\mu) \subset \Delta(\mathbb{D}([0,1],\mathbb{R}^d))$ of martingales whose law at time 1 is Blackwell dominated by μ is also compact. Moreover, the M-Z topology coincides with the product M-Z-topology on $\mathbb{D}([0,1],\mathbb{R})^d$ (which is not the case for the Skorokhod topology).*

PROOF. The topology introduced in [**45**] was defined on $\mathbb{D}([0,\infty),\mathbb{R}^d)$ and the definition given above is just the induced topology on $\Delta(\mathbb{D}([0,1],\mathbb{R}^d))$ which is seen as the closed subset of functions that remain constant after time 1. The first result is therefore the traduction of theorem 2 in [**45**]. The second follows from the fact that the projection $(X_t)_{t\in[0,1]} \to X_1$ at time 1 is (M-Z)-continuous and that the condition $[X_1] \leq \mu$ is closed. The last is obvious from the definition. $\square$

It is now quite direct from the preceding results to express the problem as an optimization problem over continuous-time martingales :

LEMMA 9.2:

$$V_\infty(\mu) = W(\mu) \triangleq \max_{[(X_t)_{t\in[0,1]}]\in\mathcal{M}(\preceq_\mu)} H([(X_t)_{t\in[0,1]}]) \tag{9.1}$$

where

$$H(\mathbb{P}) \triangleq \max_{[(X_t,Z_t)_{t\in[0,1]}]\in\mathcal{M}(\mathbb{P},Q_\Gamma)} \mathbb{E}[\langle X_1, Z_1\rangle] \tag{9.2}$$

and where $\mathcal{M}(\mathbb{P},Q_\Gamma)$ is the set of martingales distributions in $\Delta(\mathbb{D}([0,1],\mathbb{R}^{2d}))$ of processes (X,Z) such that $[(X_t)_{t\in[0,1]}] = \mathbb{P}$ and $[(Z_t)_{t\in[0,1]}] \in Q_\Gamma$ (using the identification of the continuous functions as a subset of $\mathbb{D}$). The set of maximizers is a non empty (M-Z)-compact convex subset of $\mathcal{M}(\preceq_\mu)$ denoted $\mathcal{P}_\infty(\mu)$.

PROOF. From the definition of V_∞, we have

$$V_\infty(\mu) = \max_{[(Z_t)_{t\in[0,1]}]\in Q_\Gamma, [L]\preceq\mu} \mathbb{E}[\langle L, Z_1\rangle]$$

Therefore

$$W(\mu) = \max_{[(X_t,Z_t)_{t\in[0,1]}]\in\mathcal{M}(\preceq_\mu,Q_\Gamma)} \mathbb{E}[\langle X_1, Z_1\rangle] \leq V_\infty(\mu)$$

where $\mathcal{M}(\preceq_\mu, Q_\Gamma) = \bigcup_{\mathbb{P}\in\mathcal{M}(\preceq_\mu)} \mathcal{M}(\mathbb{P}, Q_\Gamma)$ since the marginal distribution of $[X_1,(Z_t)_{t\in[0,1]}]$ fulfills the constraints of the definition.

For the converse inequality, just define $X_t = \mathbb{E}[L \mid Z_s,\ s \leq t]$. The set $\mathcal{M}(\preceq_\mu, Q_\Gamma)$ is (M-Z)-compact since it is the intersection of the set of martingale distributions uniformly bounded in L^2 by $(C_\Gamma + \|\mu\|_2)$ and of the set $\mathcal{P}(\mathcal{M}(\preceq_\mu), Q_\Gamma)$. Compactness and convexity of $\mathcal{P}(\mathcal{M}(\preceq_\mu), Q_\Gamma)$ follows from lemma 12.1. Indeed, the M-Z topology is a product topology and it is weaker than the Skorokhod topology (so that Q_Γ is M-Z compact). The application

$$[(X_t, Z_t)_{t\in[0,1]}] \longrightarrow \mathbb{E}[\langle X_1, Z_1\rangle]$$

is (M-Z)-continuous and affine on $\mathcal{M}(\preceq_\mu, Q_\Gamma)$ since the projection at time 1 is linear and continuous and using lemma 12.3. We deduce that the set of maximizers is nonempty and compact convex. Its marginal projection $\mathcal{P}_\infty(\mu)$ on the first coordinate of the product $\mathbb{D}([0,1],\mathbb{R}^d)^2$ is then compact convex. □

Proof of theorem 1.2. Using the proof of lemma 4.1, given an optimal sequence of martingales $((L^n_k)_{k=1,..,n})_{n\in\mathbb{N}}$, we can construct a sequence $((L^n_k, S^n_k)_{k=1,..,n})_{n\in\mathbb{N}}$ such that $[(S^n_k)_{k=1,..,n}] \in T^n$ and

$$\frac{\mathcal{V}^V_n((L_k)_{k=1,..,n})}{\sqrt{n}} \leq \mathbb{E}[\langle L^n_n, \frac{\sum_{k=1}^n S^n_k}{\sqrt{n}}\rangle]$$

Denoting the continuous-time versions

$$Z^n_t = \frac{1}{\sqrt{n}}\sum_{k=1}^{\lfloor nt\rfloor} S^n_k, \qquad X^n_t = L^n_{\lfloor nt\rfloor}$$

the sequence of joint distributions $[(X^n_t, Z^n_t)_{t\in[0,1]}]$ is (M-Z)-relatively compact from lemma 9.1. Any limit distribution is a martingale using that the sets of uniformly L^2 bounded martingale's distributions are closed. The marginal laws of the coordinates of any limiting distribution are respectively in the compact sets $\mathcal{M}(\preceq_\mu)$ by lemma 9.1 and Q_Γ using proposition 5.1 (convergence to an element of Q_Γ holds for a stronger topology along a subsequence). Moreover, along any convergent subsequence we know from lemma 12.3 that the application

$$[(X^n_t, Z^n_t)_{t\in[0,1]}] \longrightarrow \mathbb{E}[\langle X^n_1, Z^n_1\rangle]$$

is continuous and since by hypothesis $\mathbb{E}[\langle X^n_1, Z^n_1\rangle] \underset{n\to\infty}{\longrightarrow} V_\infty(\mu)$, we deduce that the limiting distribution of $[(X^n_t)_{t\in[0,1]}]$ belongs to $\mathcal{P}_\infty(\mu)$. □

10. Asymptotic distributions : Characterization

In order to obtain to study the properties of the set $\mathcal{P}_\infty$, we introduce the time-dependent value function.

Definition 10.1:

$$U(t,\mu) = \sup_{\nu\in Q_\Gamma(t)} C(\mu,\nu)$$

Lemma 10.1: *For all $t\in[0,1]$ and $\mu\in\Delta^2$, we have*

$$U(t,\mu) = \sqrt{t}V_\infty(\mu)$$

Proof. If $(Z_t)_{t\geq 0}$ is a martingale, then $Y_t = (\alpha^{-\frac{1}{2}}Z_{\alpha t})$ is a martingale such that $\langle Y\rangle_t = \alpha^{-1}\langle Z\rangle_{\alpha t}$. It follows easily that $Q_\Gamma(t) = (x\to\sqrt{t}x)\sharp Q_\Gamma(1)$ and we conclude the proof using that $\nu\to C(\mu,\nu)$ is positively homogenous in the sense (A3). □

Lemma 10.2: *Let $\mu_1 \preceq \mu_2 \in \Delta^2$ and let (X_1, X_2) a martingale such that $X_i\sim\mu_i$ for $i=1,2$. Then, for all $t\in[0,1]$,*

$$V_\infty(\mu_2) \geq U(t,\mu_1) + \mathbb{E}[U(1-t,[X_2 \mid X_1])]$$

PROOF. Note at first that the cases $t = 0$ and $t = 1$ follows respectively from Jensen's inequality (lemma 5.1 in chapter 1) and from the Blackwell nondecreasing property of V_∞ (which is a supremum of nondecreasing functions). Let $t \in (0,1)$ and $(\widehat{X}_1, (Z_s)_{s\in[0,t]})$ be optimal for the problem $U(t, \mu_1)$, which means

$$[\widehat{X}_1] = \mu_1, \; [(Z_s)_{s\in[0,t]}] \in \pi_t(Q_\Gamma), \text{ and } U(t, \mu_1) = \mathbb{E}[\langle \widehat{X}_1, Z_t\rangle].$$

Let $F(x)$ be a regular version of the conditional law of X_2 given $X_1 = x$ and let Ψ be a measurable selection on Δ^2 of the set-valued mapping (see theorem 2.1 in the appendix)

$$\mu \to \underset{\pi\in\mathcal{P}(\mu,\pi_{1-t}(Q_\Gamma))}{argmax} \int \langle x, y(1-t)\rangle d\pi(x, y(.))$$

Construct on an enlarged probability space a variable $(\widehat{X_2}, (Y_u)_{u\in[0,1-t]})$ whose conditional law given $(\widehat{X}_1, (Z_s)_{s\in[0,t]})$ is $\Psi(F(X_1))$. Using lemma 8.2, the law of the process $\widehat{Z}_s = Z_{s\wedge t} + Y_{(s\vee t)-t}$ is in Q_Γ and

$$\begin{aligned} U(t, \mu_1) + \mathbb{E}[U(1-t, [X_2 \mid X_1])] &= \mathbb{E}[\langle \widehat{X}_1, Z_t\rangle] + \mathbb{E}[\mathbb{E}[\langle \widehat{X_2}, Y_{1-t}\rangle \mid \widehat{X}_1, (Z_s)_{s\in[0,t]}]] \\ &= \mathbb{E}[\langle \widehat{X_2}, \widehat{Z_1}\rangle] \le V_\infty(\mu_2) \end{aligned}$$

□

LEMMA 10.3: *For all (law of) martingale X in $\mathcal{P}_\infty(\mu)$, we have*

$$V_\infty(\mu) = U(t, [X_t]) + \mathbb{E}[U(1-t, [X_1 - X_t \mid X_t])]$$

PROOF. Using lemma 9.2, for any law in $\mathcal{P}_\infty(\mu)$, there exists a (law of) martingale $(X_t, Z_t)_{t\in[0,1]}$ maximizing H (see (9.2)) in $\mathcal{M}(\preceq_\mu, Q_\Gamma)$ such that X follows the prescribed law. Note at first that the martingale property implies

$$V_\infty(\mu) = \mathbb{E}[\langle X_1, Z_1\rangle] = \mathbb{E}[\langle X_t, Z_t\rangle] + \mathbb{E}[\langle X_1 - X_t, Z_1 - Z_t\rangle]$$

Assume that $\mathbb{E}[\langle X_t, Z_t\rangle] < U(t, [X_t])$. Let $F(x)$ be a regular version of the conditional law of $(X_1 - X_t, (Z_s - Z_t)_{s\ge t})$ given X_t. F is almost surely valued in $\mathcal{P}(\Delta^2, \pi_{1-t}(Q_\Gamma))$ using lemma 8.2. Let $(S, (W_s)_{s\in[0,t]})$ be a pair such that $[S] = [X_t]$, $[(W_s)_{s\in[0,t]}] \in \pi_t(Q_\Gamma)$ and $\mathbb{E}[\langle S, W_t\rangle] = U(t, [X_t])$. Construct on a possibly enlarged probability space a pair $(T, (Y_s)_{s\in[0,1-t]})$ whose conditional law given $(S, (W_{s\in[0,t]}))$ is $F(S)$. It follows that $[S + T] = \mu$, $[(\widehat{Z}_s)_{s\in[0,1]}] \in Q_\Gamma$ with $\widehat{Z}_s = W_{s\wedge t} + Y_{(s\vee t)-t}$ (using lemma 8.2 and the fact that $\pi_{1-t}(Q_\Gamma)$ is a closed convex set) and

$$\mathbb{E}[\langle S + T, \widehat{Z}_1\rangle] = \mathbb{E}[\langle S, W_t\rangle] + \mathbb{E}[\langle T, Y_{1-t}\rangle] > V_\infty(\mu)$$

which contradicts the definition of $V_\infty(\mu)$. The second part of the proof is similar to the previous lemma. □

LEMMA 10.4: *The function V_∞ is strictly increasing with respect to the Blackwell order.*

PROOF. Let $\mu_1 \preceq \mu_2 \in \Delta^2$ and let (X_1, X_2) a martingale such that $X_i \sim \mu_i$ for $i = 1, 2$. Assume that $V_\infty(\mu_1) = V_\infty(\mu_2)$. From the previous lemma, we have for all $t \in (0,1)$

$$V_\infty(\mu_2) \ge U(t, \mu_1) + \mathbb{E}[U(1-t, [X_2 \mid X_1])] = \sqrt{t}V_\infty(\mu_1) + \sqrt{1-t}\mathbb{E}[V_\infty([X_2 \mid X_1])]$$

This implies

$$\mathbb{E}[V_\infty([X_2 \mid X_1])] \leq \frac{1-\sqrt{t}}{\sqrt{1-t}} V_\infty(\mu_2)$$

and we deduce that the first term is equal to zero by sending t to 1. In order to conclude that $\mu_1 = \mu_2$, it remains to prove that $V_\infty(\mu) = 0$ implies that μ is a Dirac mass. Recall that $V_\infty(\mu) = \sup_{\nu \in Q_\Gamma(1)} C(\mu, \nu)$. Using lemma 12.4, it is then sufficient to prove that $Q_\Gamma(1)$ contains a law which is absolutely continuous with respect to Lebesgue's measure. But Q_Γ contains the laws of Brownian motion processes with constant instantaneous covariance equal to $P \in \Gamma \cap S^d_{++}$ which concludes the proof. $\square$

Notation 9: *Given some function $\phi \in Conv(\mathbb{R}^d)$, and some $\mathbb{R}^d$-valued random variables Z, X, then*

$$X = \nabla\phi(Z) \text{ means } \quad \begin{cases} \mathbb{P}(Z \in \{x \in \mathbb{R}^d \,:\, \nabla\phi \text{ exists}\}) = 1 \\ \mathbb{P}(X \in \partial\phi(Z)) = 1 \end{cases}$$

In this case, the random variable $\nabla\phi(Z)$ is well-defined and is almost surely equal to any $g(Z)$ for g a measurable selection of the subdifferential of ϕ.

Lemma 10.5: *Let $\mu \in \Delta^2$ and $\phi \in \partial V_\infty(\mu)$. Then for any optimal joint distribution of (L, Z_1) in the problem $V_\infty(\mu)$, we have $L = \nabla\phi^*(Z_1)$ almost surely.*

Proof. Using lemma 7.1, for any optimal variables (L, Z_1) we have $L \in \partial\phi^*(Z_1)$ almost surely. It follows that

$$V_\infty(\mu) = \mathbb{E}[\langle L, Z_1\rangle] = \mathbb{E}[\langle g(Z_1), Z_1\rangle] = V_\infty(\hat{\mu})$$

where μ is the law of L and $\hat{\mu}$ the law of $g(Z_1) = \mathbb{E}[L \mid Z_1]$ since $g(Z_1) \in \partial\phi^*(Z_1)$ using the properties of the subgradient. On the other hand, $\hat{\mu}$ is Blackwell dominated by μ. Using lemma 10.4, V_∞ is strictly increasing and therefore $\mu = \hat{\mu}$ which implies $L = g(Z_1)$. To conclude define the variable X such that its conditional law given Z_1 is uniform on the set [8] $(g(Z_1) + B(0, \varepsilon)) \cap \partial\phi^*(Z_1)$ and a Dirac mass on $\nabla\phi^*(Z_1)$ when this set is reduced to a single point. $X \in L^2$ and $\widehat{X} = \mathbb{E}[X \mid Z_1] \in L^2$ since $|X - L| \leq \varepsilon$. Applying again lemmas 7.1 and 10.4, we deduce as above that $X = \widehat{X}$ almost surely, which implies $g(Z_1) = \nabla\phi^*(Z_1)$ and concludes the proof. $\square$

Proposition 10.1: *Let $\mu \in \Delta^2$ and $\phi \in \partial V_\infty(\mu)$. Then for any (law of) martingale (X, Z) in $\mathcal{M}(\preceq_\mu, Q_\Gamma)$ maximizing H, we have for all $t \in [0, 1]$*

$$X_t = \nabla u(t, Z_t) \quad \text{almost surely.}$$

where u is the solution of (8.1).

Proof. Let us fix $t \in [0, 1]$. Using lemma 10.5, we have

$$X_t = \mathbb{E}[X_1 \mid \mathcal{F}_t^{X,Z}] = \mathbb{E}[\nabla\phi^*(Z_1) \mid \mathcal{F}_t^{X,Z}].$$

8. The probability whose density with respect to the Lebesgue's measure on the affine subspace generated by this convex set, is just the normalized indicator function of the set.

But from lemma 10.3, we know that $\mathbb{E}[\langle X_t, Z_t\rangle] = U(t,[X_t])$. Using then lemma 10.1, 7.1 and 10.5, we deduce that $X_t = \nabla g(Z_t)$ almost surely for some function $g \in Conv(\mathbb{R}^d)$. It follows that

$$X_t = \mathbb{E}[X_1 \mid Z_t] = \mathbb{E}[\nabla\phi^*(Z_1) \mid Z_t].$$

By lemma 7.1, the process Z is optimal in the dual problem which implies that $u(t, Z_t) = \mathbb{E}[\phi^*(Z_1) \mid Z_t]$. Let v denotes a measurable selection of $\partial\phi^*$. We have by definition

$$\forall z, h \in \mathbb{R}^d,\ \phi^*(z+u+h) \geq \phi^*(z+u) + \langle v(z+u), h\rangle$$

Taking conditional expectations, we obtain

$$\begin{aligned} u(t, Z_t + h) &= \mathbb{E}[\phi^*(Z_t + (Z_1 - Z - t) + h) \mid Z_t] \\ &\geq \mathbb{E}[\phi^*(Z_t + (Z_1 - Z_t)) \mid Z_t] + \langle \mathbb{E}[v(Z_t + (Z_1 - Z_t)) \mid Z_t], h\rangle \end{aligned}$$

Note that since $X_1 = v(Z_1)$, $X_t = \mathbb{E}[v(Z_t + (Z_1 - Z_t)) \mid Z_t]$ and it follows that

$$u(t, Z_t + h) \geq u(t, Z_t) + \langle X_t, h\rangle$$

We conclude that $X_t \in \partial u(t, Z_t)$ since the above inequality holds almost surely for a countable dense subset of h in $\mathbb{R}^d$. The end of the proof is similar to lemma 10.5. □

We can now state a corollary, which is a kind of verification theorem.

Theorem 10.1: *Under the same hypotheses as proposition 10.1 and if the solution u is C^1 with respect to the space variable, then $\mathcal{P}_\infty$ is the set of all laws of processes*

$$(X_t)_{t\in[0,1]} = (\nabla u(t, Z_t))_{t\in[0,1]}$$

where the law process Z ranges through the set of maximizers of $V^*_\infty(\phi)$ such that $\nabla\phi^*(Z_1) \sim \mu$.

Proof. For any (law of) martingale (X, Z) in $\mathcal{M}(\preceq_\mu, Q_\Gamma)$ maximizing H, we have from proposition 10.1 with probability 1,

$$\forall t \in [0,1],\ t \text{ rational}, \quad (X_t)_{t\in[0,1]} = (\nabla u(t, Z_t))_{t\in[0,1]}$$

The process in the right-hand side has continuous trajectories and X has càdlàg trajectories so that the equality can be extended to all $t \in [0,1]$. The results follows then from proposition 10.1. □

Let us now prove the result announced in the introduction.

Proof of theorem 1.3. In view of the previous results, we only need to prove that Z is a maximizers of $V^*_\infty(\phi)$ if and only if property (1.6) is true. But this follows directly from Ito's formula since u is assumed to be $C^{1,2}$. □

An example of explicit solution using this method is given in chapter 3.

We state now some easy properties that allows to reduce the size of Γ.

Definition 10.2: *Let $\partial^+\Gamma$ be positive boundary of Γ in the following sense*

$$\partial^+\Gamma = \{M \in S^d_+ \,:\, (M + S^d_{++}) \cap \Gamma = \emptyset\}$$

DEFINITION 10.3: *Let $Q_{\partial^+\Gamma}$ be the set of laws $\mathbb{P}$ in Q_Γ such that*

$$\mathbb{E}_{\mathbb{P}}[\int_0^1 \mathbb{1}_{\rho_s \notin \partial^+\Gamma} ds] = 0$$

where ρ is the process defined by

$$\rho_s = \overline{lim}_n n(\langle Z\rangle_s - \langle Z\rangle_{(s-\frac{1}{n})^+})$$

and Z denotes the canonical coordinate process.

LEMMA 10.6: *Let f be a real-valued convex function defined on $\mathbb{R}^d$ such that*

$$\sup_{\nu \in Q_\Gamma(1)} \int f d\nu < \infty.$$

For all $\mathbb{P} \in Q_\Gamma$ there exists $\mathbb{P}^+ \in Q_{\partial^+\Gamma}$ such that if Z and Z^+ follow respectively the laws $\mathbb{P}$ and $\mathbb{P}^+$, we have

$$\mathbb{E}_{\mathbb{P}}[f(Z_1)] \leq \mathbb{E}_{\mathbb{P}^+}[f(Z_1^+)]$$

Moreover, if $\mathbb{P} \notin Q_\Gamma^+$ and f is non-linear, this inequality is strict.

PROOF. Let Z be defined on the canonical space. Let us consider an extension of the filtered probability space on which is defined a d-dimensional standard Brownian motion W independent of Z. From the definition of Q_Γ, the process ρ given in definition 10.3 has values in Γ with probability 1 and

$$\langle Z\rangle_t = \int_0^t \rho_s ds$$

Let $h : \Gamma \to \partial^+\Gamma$ be a measurable function such that $h(M) - M \in S_{++}^d$ if $M \notin \partial^+\Gamma$ and $h(M) = M$ otherwise. Define then

$$Y_t = \int_0^t (h(\rho_s) - \rho_s)^{\frac{1}{2}} dW_s$$

The quadratic covariation process $\langle Z, Y\rangle = 0$ by construction and $Z^+ = Z + Y$ has a law in $Q_{\partial^+\Gamma}$. Using the properties of the Wiener integral, since W is independent from Z, we have that the conditional law of Z_1^+ given $\mathcal{F}_1^Z$ is a gaussian distribution with mean Z_1 and with covariance matrix $\int_0^1 (h(\rho_s) - \rho_s) ds$ (see proposition 1.1 in [**35**]). Therefore (Z_1, Z_1^+) is a two-step martingale and the inequality follows from Jensen's inequality. If moreover the law of Z is not in $Q_{\partial^+\Gamma}$, then this implies that $\mathbb{P}(\int_0^1 (h(\rho_s) - \rho_s) ds \in S_{++}^d) > 0$ and we deduce that $\mathbb{E}[f(Z_1^+) \mid \mathcal{F}_1^Z] > f(Z_1)$ on this set which in turn implies the result. □

Note that $\partial^+\Gamma$ is closed but not necessarily convex so that $Q_{\partial^+\Gamma}$ is not closed in general. We will see however two examples in the next section where this set is convex. As a corollary, we have

COROLLARY 3: *For all $\mu \in \Delta^2$*

$$V_\infty(\mu) = \sup_{\nu \in Q_{\partial^+\Gamma}(1)} C(\mu, \nu)$$

PROOF. The last lemma implies that laws in $Q_\Gamma(1)$ are always Blackwell dominated by some law in $Q_{\partial^+\Gamma}(1)$. The conclusion foolows using the nondecreasing property of C. □

11. Application : the unidimensional case and the L^1-variation

Our examples will be based on the following simple type of L^2-convex subset of S_+^d :

Lemma 11.1: *If Φ is a comprehensive convex subset of S_+^d, i.e. such that*

$$N \in \Phi \text{ and } M \leq N \Rightarrow M \in \Phi \tag{11.1}$$

then Φ is L^2-convex.

Proof. Let $X, Y \in \widehat{\Phi}$ and $P = cov(X)$ $Q = cov(Y)$. Then

$$cov(\lambda X + (1-\lambda)Y) = \lambda^2 P + (1-\lambda)^2 Q + \lambda(1-\lambda)(cov(X,Y) + cov(Y,X))$$

Since $\lambda P + (1-\lambda)Q \in \Phi$, we just have to show that the difference is nonnegative

$$\begin{aligned}\lambda P + (1-\lambda Q) - \Big(\lambda^2 P + (1-\lambda)^2 Q + \lambda(1-\lambda)(cov(X,Y) + cov(Y,X))\Big)\\ = \lambda(1-\lambda)(P + Q - cov(X,Y) - cov(Y,X)) = \lambda(1-\lambda)cov(X-Y) \geq 0\end{aligned}$$

□

The one-dimensional problem.

Assume that $d = 1$ and assumptions A1-A4. Then

Theorem 11.1: *For all $\mu \in \Delta^2(\mathbb{R})$, the set $\mathcal{P}_\infty(\mu)$ is reduced to the single point $\mathbb{P}^\mu$ which is the law of the martingale*

$$X_t^\mu = \mathbb{E}[f_\mu(B_1) \mid \mathcal{F}_t^B]$$

where B is a standard Brownian motion and

$$f_\mu = F_\mu^{-1} \circ F_{\mathcal{N}(0,\rho^2)}$$

where $\rho = r(1)$ and F_μ^{-1} is the right-continuous inverse of the distribution function of μ.

Proof. Since r is nondecreasing, the set $F = \{r \leq 1\}$ is an interval $[0, 1/\rho^2]$ where ρ^2 is such that $r(1) = \rho$. The set $\widehat{F}$ is the closed ball in L_0^2 of radius $1/\rho$, and this implies A5 by lemma 11.1. Now $\widehat{G}$ is the ball of radius ρ and therefore $G = \Gamma$ and Q_Γ is the set of distributions of continuous $\mathbb{R}$-valued martingales such that $\langle Z \rangle$ is ρ^2-Lipschitz with respect to the time-variable. Proposition 7.1 implies

$$V_\infty(\mu) = \min_{\phi \in Conv(\mathbb{R})} \Big(\int \phi d\mu + \sup_{\nu \in Q_\Gamma(1)} \int \phi^* d\nu\Big)$$

We assume that the optimal ϕ is such that ϕ^* is not linear, otherwise, μ would be a Dirac mass, and the solution in this case is obvious. For any such ϕ, the law of a Brownian motion with instantaneous variance ρ^2 is the unique maximizer of

$$\sup_{[Z_t]_{t \in [0,1]}} \mathbb{E}[\phi^*(Z_1)]$$

using proposition 10.6, and since $\partial^+\Gamma = \{\rho^2\}$. It remains to characterize the optimal ϕ. Using the previous result and proposition 7.1, if ϕ is optimal for μ, then $\mu = \nabla\phi^* \sharp \mathcal{N}(0,\rho)$. This relation implies that ϕ^* is given by

$$\phi^*(x) = \int_0^x f_\mu(u)du \quad \text{with} \quad f_\mu = F_\mu^{-1} \circ F_{\mathcal{N}(0,\rho^2)}$$

We conclude finally that $\mathcal{P}_\infty(\mu) = \{\mathbb{P}(\mu)\}$ with $\mathbb{P}^\mu$ the law of the martingale

$$X_t^\mu = \mathbb{E}[f_\mu(B_1) \mid \mathcal{F}_t^B]$$

To prove this, note that any maximizer (X,B) of H in $\mathcal{M}(\mu, Q_\Gamma)$ is a martingale, and optimality implies that B is an $\mathcal{F}^{X,B}$-Brownian motion and $X_1 = f_\mu(B_1)$. Therefore,

$$X_t = \mathbb{E}[X_1 \mid \mathcal{F}_t^{X,B}] = \mathbb{E}[f_\mu(B_1) \mid \mathcal{F}_t^B] = u(t, B_t)$$

where u is the solution of the backward heat equation with terminal condition $u(1,x) = f_\mu(x)$. □

Remark 11.1: *Using this result and theorem 1.2, we recover the main result given in* [**25**] *where* X^μ *is called the Continuous Martingale of Maximal Variation with final distribution* μ.

Symmetric separable functions and the L^1-norm. Consider functions V given by

$$V(\mu) = \sum_{i=1}^d w(\mu_i)$$

where μ_i are the marginal distributions of μ and w is a function defined on $\Delta^2(\mathbb{R})$ meeting assumptions A1-A4. Let us add the following symmetry hypothesis :

$$\forall X_i \in L^2,\ w([-X_i]) = w([X_i])$$

Theorem 11.2: *Under the preceding hypotheses, the assumption A5 is true, and*

$$\partial^+\Gamma = co(\{\rho^2 (M_{ij})_{i,j=1,..,d} \,:\, \forall i,\ M_{ii} = 1,\ \forall i<j,\ M_{ij} \in \{-1,1\}\})$$

with $\rho = \underset{\mathbb{E}[X_i^2]=1}{sup}\ w([X_i])$.

Proof. From the definition of r we have

$$r(M) = \underset{cov(X)=M}{sup} (\sum_{i=1}^d w([X_i])) \leq \sum_{i=1}^d \underset{\mathbb{E}[X_i^2]=M_{ii}}{sup} w([X_i]) = \sum_{i=1}^d \rho\sqrt{M_{ii}}$$

where $\rho = \underset{\mathbb{E}[X_i^2]=1}{sup}\ w([X_i])$. The symmetry assumption implies that the above inequality is an equality. At first, since w is concave, for any random variable $X_i \in L^2$, the law obtained by the convex combination $\frac{1}{2}([X_i] + [-X_i])$ in $\Delta^2(\mathbb{R})$ is such that

$$w(\frac{1}{2}([X_i] + [-X_i])) \geq w([X_i])$$

Therefore we can restrict the supremum in the definition of ρ to the set of symmetric distribution, i.e. laws of variables X_i such that $[X_i] = [-X_i]$. Let (ν) be an ε-optimal symmetric law for this supremum. Let D_ν be the set of attainable covariance matrices of vectors $X = (X_i)_{i=1,..,d}$ such that $[(M_{ii})^{-1/2}X_i] = \nu$ for all $i = 1, .., d$. For such a vector X, we have

$$\sum_{i=1}^{d} w([X_i]) \geq \sum_{i=1}^{d} (\rho - \varepsilon)\sqrt{M_{ii}}$$

The result follows then by sending ε to zero if we can prove that $M \in D_\nu$. D_ν is a closed convex set included in the convex set D of S_+^d of matrices with diagonal $(M_{ii})_{i=1,..,d}$. This inclusion is an equality since D_ν contains extremal elements of D that are obtained by variables of the form $(X_1, \pm X_1, .., \pm X_1)$. Finally, the set $\widehat{F}$ is given by

$$\widehat{F} = \{X \in L^2 \, : \, \rho \sum_{i=1}^{d} \|X_i\|_{L^2} \leq 1\}$$

which clearly implies A5 by lemma 11.1. We check easily that

$$\widehat{G} = \{X \in L^2 \, : \, \max_{i=1,..,n} \|X_i\|_{L^2} \leq \rho\}$$

The corresponding set G is the set of matrices $M \in S_+^d$ such that for all $i = 1, .., d$, $M_{ii} \leq \rho^2$. Since it is already convex, we have $\Gamma = G$ and

$$\partial^+\Gamma = co(\{\rho^2 (M_{ij})_{i,j=1,..,d} \, : \, \forall i, \, M_{ii} = 1, \; \forall i < j, \, M_{ij} \in \{-1, 1\}\})$$

□

Remark 11.2: *This result applies in particular for $w = \|.\|_{L^1}$ with $\rho = 1$ and will be applied in chapter 3 to financial games. Note also that the set $Q_{\partial^+\Gamma}$ in the previous result is exactly the set of martingales $(Z^1, .., Z^d)$ such that any coordinate is a Brownian motion with covariance ρ^2.*

12. Auxiliary results and technical proofs.

12.1. Optimal transportation and Wasserstein distances. We present in this section results about optimal transportation and Wasserstein distances. This material is well-known and can be found in [**57**] or [**2**]. Let X,Y be two separable metric spaces and Q,R two subsets of $\Delta(X)$ and $\Delta(Y)$. Then $\mathcal{P}(Q, R)$ denotes the set of probabilities over $X \times Y$ whose marginal distributions over X and Y belongs respectively to Q and R. If $Q = \{\mu\}$, we will simply write $\mathcal{P}(\mu, R)$.

Lemma 12.1: *Let X,Y be two separable metric spaces and Q,R two tight (resp. closed, convex) subsets of $\Delta(X)$ and $\Delta(Y)$. Then the set $\mathcal{P}(Q, R)$ is itself tight (resp. closed, convex).*

The Wasserstein distances. The Wasserstein distance of order p is defined on the set $\Delta^p(\mathbb{R}^d)$ by

$$d_{W_p}(\mu, \nu) = \min_{\pi \in \mathcal{P}(\mu,\nu)} (\int |y - x|_p^p d\pi(x, y))^{\frac{1}{p}} = min\{\|X - Y\|_{L^p} \mid X \sim \mu \, , \, Y \sim \nu\}$$

The metric space $(\Delta^p(\mathbb{R}^d), d_{W_p})$ is polish. Convergence for d_{W_p} is equivalent to classic weak convergence together convergence of the moments of order p and the sets of probabilities with uniformly integrable moments of order p are relatively compact. Moreover, we have the following useful lemma

LEMMA 12.2: *For any continuous function f and $K > 0$ such that $| f(x) | \leq K(1+ | x |^p)$, the application*

$$\Delta^p(\mathbb{R}^d) \to \mathbb{R} : \pi \to \int f(x) d\pi(x)$$

is d_{W_p}- continuous

and the following which is lemma 5.2.4 in [**2**].

LEMMA 12.3: *Let $X = Y = \mathbb{R}^d$ and $\pi_n \in \mathcal{P}(X \times Y)$ be a weakly converging sequence with limit π such that*

$$\sup_n \int |x|^p + |y|^q d\pi_n(x,y) < \infty \quad \textit{for some } \ p,q \in (1,\infty) \textit{ such that } \frac{1}{p} + \frac{1}{q} = 1$$

If the sequence of marginals μ_n on X has uniformly integrable moments or order p (resp. ν_n on Y has uniformly integrable moments of order q) then

$$\int \langle x,y \rangle d\pi_n(x,y) \underset{n\to\infty}{\longrightarrow} \int \langle x,y \rangle d\pi(x,y)$$

Maximal covariance functions. . These functions are also optimal transport value functions, related to the square Wasserstein distance. Precisely, the maximal covariance between two probabilities on $\mathbb{R}^d$ is defined by

$$C : \Delta^2(\mathbb{R}^d) \times \Delta^2(\mathbb{R}^d) \longrightarrow \mathbb{R} : (\mu,\nu) \longrightarrow \max_{\pi \in P(\mu,\nu)} \int \langle x,y \rangle d\pi(x,y)$$

We have then the straightforward relation

$$\forall \mu,\nu \in \Delta^2(\mathbb{R}^d), \qquad d^2_{W_2}(\mu,\nu) = \|\mu\|_2^2 + \|\nu\|_2^2 - 2C(\mu,\nu)$$

and the classical dual equality

THEOREM 12.1: *For all $\mu,\nu \in \Delta^2(\mathbb{R}^d)$, we have the following equalities*

$$\begin{aligned}\max_{\pi \in \mathcal{P}(\mu,\nu)} \int \langle x,y \rangle d\pi(x,y) &= \inf_{(\phi-\frac{1}{2}|.|^2, \psi-\frac{1}{2}|.|^2) \in C_b(\mathbb{R}^d)^2; \phi+\psi \geqslant \langle .,. \rangle} \left(\int \phi d\mu + \int \psi d\nu\right) \\ &= \min_{\phi \in Conv(\mathbb{R}^d)} \int \phi d\mu + \int \phi^* d\nu\end{aligned}$$

where $\phi + \psi \geqslant \langle .,. \rangle$ means $\phi(x) + \psi(y) \geqslant \langle x,y \rangle$ for all x,y and $C_b(\mathbb{R}^d)$ denotes the set of real-valued bounded continuous functions on $\mathbb{R}^d$.

Let us also mention the following characterization

THEOREM 12.2: *For all $\mu,\nu \in \Delta^2(\mathbb{R}^d)$, we have the following equivalence*

$$\pi^* \in \operatorname*{argmax}_{\pi \in \mathcal{P}(\mu,\nu)} \int \langle x,y \rangle d\pi(x,y) \iff \exists \phi \in Conv(\mathbb{R}^d), \quad y \in \partial\phi(x) \quad \pi^* \textit{-almost surely}$$

where $\partial\phi$ denotes the subgradient.

Blackwell order.

DEFINITION 12.1: *$\mu_1 \Delta(\mathbb{R}^d)$ is Blackwell-dominated by $\mu_2 \in \Delta(\mathbb{R}^d)$ (denoted $\mu_1 \preceq \mu_2$) if there exists a two-steps martingale X_1, X_2 such that X_i is μ_i distributed for $i = 1, 2$.*

This order is also often called convex order between probability measures since (see Blackwell [**12**])

$$\mu_1 \preceq \mu_2 \Leftrightarrow \forall f \in Conv(\mathbb{R}^d), \int f d\mu_1 \leq \int f d\mu_2 \tag{12.1}$$

Let us now list some useful properties

LEMMA 12.4: *The set $\{\nu \in \Delta^2 : \nu \preceq \mu\}$ is d_{W_2}-compact (hence weakly compact). The function $\mu \to C(\mu, \nu)$ is nondecreasing for the Blackwell order, strictly if ν is absolutely continuous with respect to the Lebesgue's measure.*

PROOF. For the first assertion, uniform integrability of the second order moment follows from jensen inequality and the martingale characterization of the Blackwell order. Closedness follows from the convex representation (12.1) since the map $\mu \to \int f d\mu$ is lower continuous for any $f \in Conv(\mathbb{R}^d)$. For the second assertion, let ν be absolutely continuous and $\mu_1 \preceq \mu_2 \in \Delta^2$. using theorem 12.2, we have

$$C(\mu_i, \nu) = \min_{\phi \in Conv(\mathbb{R}^d)} \int \phi d\mu_i + \int \phi^* d\nu$$

Let ϕ_2 be optimal in the above minimization problem for μ_2. If ϕ_2 is also optimal for μ_1, in which case theorem 12.2 implies $\nabla\phi_2 \sharp \nu = \mu_1 = \mu_2$ since ν is absolutely continuous. Therefore, if $\mu_1 \neq \mu_2$, ϕ_2 is not optimal for μ_1, and we deduce from (12.1) that

$$C(\mu_1, \nu) < \int \phi d\mu_1 + \int \phi^* d\nu \leq \int \phi d\mu_2 + \int \phi^* d\nu = C(\mu_2, \nu)$$

which concludes the proof. □

12.2. Technical proofs.

PROOF OF LEMMA 6.1. Using Caratheodory's theorem together a measurable selection result, we can parametrize points in Γ as follows

$$\forall Q \in \Gamma, \quad Q = \sum_{i=1}^{\frac{d(d+1)}{2}+1} \lambda_i(Q) P_i(Q)$$

where the λ_i form a convex combination and $P_i \in G$, all these functions being measurable. Let Z be the canonical process defined on the canonical space endowed with a law in Q_Γ. Then there exists on an extended filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,1]}, \mathbb{P})$ a d-dimensional Brownian motion W and an $\mathcal{F}$-progressively measurable process q_s such that $Z_t = \int_0^t q_s dW_s$ (see e.g. [**39**] theorem 3.4.2) such that with probability 1, $q_s q_s^T \in \Gamma$. Define $b_k^n = \int_{(k-2)/n}^{(k-1)/n} q_s ds$ for $k \geq 2$ and $b_1^n = 0$. The processes

$$q_s^n = \sum_{k=1}^n b_k^n \mathbb{I}_{[\frac{(k-1)}{n}, \frac{k}{n})}(s)$$

define a sequence of simple processes approximating q_s in the Hilbert space $L^2(\Omega\times[0,1], d\mathbb{P}\otimes dt)$, and such that b_1^n is constant, b_k^n is $\mathcal{F}_{(k-1)/n}$-measurable and $b_k^n(b_k^n)^T \in \Gamma$. The last property follows from the fact that the set of square root matrices of Γ in $\mathbb{M}_d$ is closed convex (the argument is the same as in lemma 11.1). Since the L^2-convergence mentioned above implies convergence in law at time 1, it is sufficient to prove that the law of $\int_0^1 q_s^n dW_s$ is in $\tilde{Q}_G(1)$. From the definition of $\tilde{Q}_G$, to show this, given a standard d-dimensional Brownian motion B defined on the canonical space, we have to construct a process $\tau \in \mathcal{H}_G$ such that the stochastic integral $\int_0^1 \tau_s dB_s$ has the same law as $\int_0^1 q_s^n dW_s$. Note that the law of $\int_0^1 q_s^n dW_s$ is determined by the law of the vector $(b_k^n \Delta_k^n W)_{k=1,..,n}$ where $\Delta_k^n W = W_{k/n} - W_{(k-1)/n}$. The conditional law of $nb_k^n\Delta_k^n W$ given $\mathcal{F}_{(k-1)/n}$ is a normal distribution with covariance matrix $c_k^n = b_k^n(b_k^n)^T \in \Gamma$. We will construct by induction the process τ and a sequence $(\widehat{c_k^n})_{k=1,..,n}$ such that $\widehat{c_k^n}$ is $\mathcal{F}^B_{(k-1)/n}$-measurable and $\widehat{c_1^n} = c_1^n$. Assume that the process τ on $[0,(k-1)/n)$ and the variables $(\widehat{c_i^n})_{i=1,..,k}$ are given and such that $(\int_0^{(k-1)/n} \tau_s dB_s, \widehat{c_k^n})$ has the same law as $(\int_0^{(k-1)/n} q_s^n dW_s, c_k^n)$. Define then τ on $[\frac{(k-1)}{n}, \frac{k}{n})$ as the piecewise constant process equal to $\sqrt{P_i(\widehat{c_k^n})}$ on the interval

$$\left[\frac{(k-1)}{n} + \frac{1}{n}\sum_{q=1}^{i-1}\lambda_q(\widehat{c_k^n})\ ,\ \frac{(k-1)}{n} + \frac{1}{n}\sum_{q=1}^{i}\lambda_q(\widehat{c_k^n})\right) \tag{12.2}$$

for $i = 1,.., \frac{d(d+1)}{2}+1$. By construction, the conditional law of $n\int_{(k-1)/n}^{(k)/n} \tau_s dB_s$ given $\int_0^{(k-1)/n} \tau_s dB_s$ is a normal distribution with covariance matrix $\widehat{c_k^n}$ and using our assumption it implies that $\int_0^{k/n} \tau_s dB_s$ has the same law as $\int_0^{k/n} q_s^n dW_s$. Next we construct a variable $\widehat{c_{k+1}^n}$, $\mathcal{F}^B_{k/n}$ measurable, such that the pair $(\int_0^{k/n} \tau_s dB_s, \widehat{c_{k+1}^n})$ has the same law as $(\int_0^{k/n} q_s^n dW_s, c_{k+1}^n)$. Using theorem 1.3 in the appendix, to construct $\widehat{c_{k+1}^n}$ with the prescribed conditional law given $\int_0^{k/n} \tau_s dB_s$, it is sufficient to have a diffuse random variable $\mathcal{F}^B_{k/n}$-measurable and independent from $\int_0^{k/n} \tau_s dB_s$. We can construct such a variable as a stochastic integral $\int_{(k-1)/n}^{k/n} \nu_s dB_s^1$ where B^1 is the first coordinate of B. For example, define ν_s as the piecewise constant process taking alternatively the values 1 and -1 on the partition of $[(k-1)/n, k/n)$ obtained by dividing each element of the partition given in (12.2) into two intervals of equal size. Usual properties of the stochastic integral against a Brownian motion show that this variable has the required properties, and we conclude the proof by induction. □

Proof of lemma 6.3. Let us define

$$D_n = \sup_{\nu\in RC^n(q,C)} d^2_{W_2}(\nu, \mathcal{N}(0, I_d))$$

Let the sequence $(S_i)_{i=1,..,n}$ be independent and identically distributed for some law $\mu \in RC^1(q,C)$, and let S_i^j denotes the j-th coordinate. Note at first that it follows from the martingale property that

$$cov(\sum_{k=1}^n \frac{S_i}{\sqrt{n}}) = I_d$$

In the next result, C_q is the universal constant of Burkholder's square function inequality for discrete martingales (cf [**14**]):

$$\begin{aligned}\mathbb{E}[| \sum_{k=1}^{n} \frac{S_i^j}{\sqrt{n}} |^q] &\leq \frac{C_q^q}{n^{q/2}}\mathbb{E}[\left(\sum_{k=0}^{n-1}(S_{k+1}^j - S_k^j)^2\right)^{q/2}] \\ &\leq \frac{n^{q/2-1}C_q^q}{n^{q/2}}\sum_{k=0}^{n-1}\mathbb{E}[| S_{k+1}^j - S_k^j |^q] \\ &\leq (2K)^q C_q^q\end{aligned}$$

Therefore, moments of order q are uniformly bounded independently of n. Recall that convergence in law together uniformly bounded moments of order $q > 2$ imply d_{W_2}-convergence. Since any maximizing sequence ν_n for D_n fulfills the classical Lindeberg condition of the the central limit theorem (theorem VII.5.2 in [**36**]) for row-wise independent triangular arrays (again, since laws in $RC^1(q, C)$ have bounded q-th order moments), we deduce that

$$D_n = d_{W_2}(\mu_n, \mathcal{N}(0, I_d)) \underset{n\to\infty}{\longrightarrow} 0$$

Moreover, the sets $RC^n(q, C)$ are d_{W_2} compact, and the last assertion follows directly from theorem 2.1 in the appendix. □

Proof of lemma 6.4. See [**39**] problem 2.5 p134 for i). The second point is only a slight modification of the same reference. We only sketch the proof for $d = 1$, the generalization is straightforward. Let $c_t = 0$ for $t < 0$, and define

$$c_t^{(n,\delta)} = \sum_{k=0}^{n-1} c_{\frac{k-\delta}{n}} \mathbb{1}_{[\frac{k}{n}, \frac{k+1}{n})}(t)$$

we will show that there exists a sequence δ_n in $[0, 1]$ such that

$$lim_{n\to\infty}\mathbb{E}[\int_0^1 (c_t - c_t^{n,\delta_n})^2 dt] = 0.$$

Define $g_n(\delta) = \mathbb{E}[[\int_0^1 (c_t - c_t^{n,\delta})^2 dt]$. It is shown in [**39**] that $\int_0^1 g_n(\delta)d\delta$ converges to zero using Fubini's theorem and i). Therefore, we can choose δ_n such that

$$g_n(\delta_n) \leq \int_0^1 g_n(\delta)d\delta$$

and this concludes the proof. □

Proof of lemma 6.5. Given $q > 2$ and $C > 0$ define

$$\begin{aligned}\Lambda &= \{\nu \in \Delta_0^2 \,:\, cov(\nu) \leq I_d\} \\ F_{q,C} &= \{\nu \in \Delta_0^2 \,:\, cov(\nu) = I_d,\ \|\nu\|_q \leq C\}\end{aligned}$$

For $M \in \mathbb{M}_d$, we have obviously $M\sharp(\Lambda) = \{\nu \in \Delta_0^2 \,:\, cov(\mu) \leq MM^T\}$ (recall that $M\sharp$ denotes the image probability by the linear map $x \to Mx$). Moreover, using lemma 3.1, we have

$$r(MM^T) = \sup_{\mu\in M\sharp(\Lambda)} V(\mu) = \sup_{\nu\in\Lambda} V(M\sharp\nu)$$

Since Λ is W_p-compact, there exists a maximum $\nu^* \in \Lambda$ (depending on M). We deduce that

$$\begin{aligned} r(MM^T) - \underset{\nu \in F_{q,C}}{sup} V(M\sharp\nu) &= V(M\sharp\nu^*) - \underset{\nu \in F_{q,C}}{sup} V(M\sharp\nu) \\ &\leq K d_{W_p}(M\sharp\nu^*, M\sharp F_{q,C}) \leq K\gamma d_{W_p}(\nu^*, F_{q,C}) \end{aligned}$$

where γ is a constant such that $|Mx|_p \leq \gamma|M||x|_p$. It is then sufficient to prove that

$$\underset{\nu \in \Lambda}{sup}\, d_{W_p}(\nu, F_{q,C}) \underset{C \to +\infty}{\to} 0.$$

Suppose on the contrary that there exists $\varepsilon > 0$ and some sequence $C_n \to \infty$ such that

$$\forall n \in \mathbb{N}, \quad \underset{\nu \in \Lambda}{sup}\, d_{W_p}(\nu, F_{q,C_n}) > \varepsilon$$

By compactness, there exists a maximizer $\mu_n \in \lambda$ for all C_n. Extracting a subsequence if necessary, we can assume that μ_n d_{W_p}-converges to $\mu \in \Lambda$.

Let X_n (resp. X) be random variables with distributions μ_n (resp. μ) defined on the same probability space such that X_n converges to X almost surely and in L^p. Define

$$Y_n = X_n \mathbb{1}_{\{|X_n|_q \leq \frac{1}{4}C_n\}} - \mathbb{E}[X_n \mathbb{1}_{\{|X_n|_q \leq \frac{1}{4}C_n\}}].$$

Then we can check that $\|Y_n\|_{L^q} \leq \frac{C_n}{2}$, $cov(Y_n) \leq I_d$ and $\|X_n - Y_n\|_{L^p} \underset{n\to\infty}{\longrightarrow} 0$. Let $P_n = I_d - cov(Y_n)$. In order to conclude the proof, it is sufficient to construct a sequence of variables Z_n independent of Y_n such that $cov(Z_n) = P_n$, $\|Z_n\|_{L^q} \leq \frac{C_n}{2}$ and $\|Z_n\|_{L^p} \underset{n\to\infty}{\longrightarrow} 0$. We would then have

$$\underset{\nu \in \Lambda}{sup}\, d_{W_p}(\nu, F_{q,C_n}) \leq \|X_n - (Y_n + Z_n)\|_{L^p} \underset{n\to\infty}{\longrightarrow} 0$$

and thus a contradiction since by construction $[Y_n + Z_n] \in F_{q,C_n}$. For the sake of completeness, let us now define such a sequence. Let $(U_1, .., U_d)$ be independent uniform random variables on $[0, 1]$, independent of the sequence X_n. For all $k \in \mathbb{N}^*$, define

$$f_k : [0,1] \to \mathbb{R} : x \to \begin{cases} 0 & if \quad x \leq 1 - \frac{1}{k} \\ k^2 x - k^2 + k & if \quad x > 1 - \frac{1}{k} \end{cases}$$

Define then $Z_n = \sqrt{P_n}((f_{k(n)}(U_i)^{1/2} sgn(U_i - \alpha_{k(n)}))_{i=1,..,d})$ with $k(n)$ the sequence defined by

$$k(n) = inf\{k \in \mathbb{N}^* \; : \; \|(f_k(U_i)^{1/2})_{i=1,..,d}\|_{L^p} \leq \frac{C_n}{2}\}$$

and $\alpha_{k(n)} \in [0, 1]$ chosen in order that $\mathbb{E}[Z_n] = 0$. This sequence has clearly the required properties and can also be used to prove the closure result in lemma 3.3.

The measurable selection exists from theorem 2.1 in the appendix using that

$$(M, \nu) \in \mathbb{M}_d \times F_{q,C} \to V(M\sharp\nu)$$

is jointly continuous when $F_{q,C}$ is endowed with the d_{W_2}-topology, and in particular compact. $\square$

PROOF OF LEMMA 6.6. With the notation $\Delta X_{k+1} = X_{k+1} - X_k$, we have

$$\begin{aligned}\left|\mathcal{V}_n^V(X,\mathcal{F}) - \mathcal{V}_n^V(Y,\mathcal{F})\right| &= \left|\mathbb{E}[\textstyle\sum_{k=0}^{n-1} V[\Delta X_{k+1} \mid \mathcal{F}_k] - V[\Delta Y_{k+1} \mid \mathcal{F}_k]]\right| \\ &\leq K\mathbb{E}\left[\textstyle\sum_{k=0}^{n-1}\mathbb{E}\left[|\Delta X_{k+1} - \Delta Y_{k+1}|_p^p \mid \mathcal{F}_k\right]^{\frac{1}{p}}\right] \\ &\leq \beta K\mathbb{E}\left[\textstyle\sum_{k=0}^{n-1}\mathbb{E}\left[\,|\Delta X_{k+1} - \Delta Y_{k+1}|^2 \mid \mathcal{F}_k^n\right]^{\frac{1}{2}}\right]\end{aligned}$$

Due to Cauchy Schwartz and Jensen's inequalities, we have

$$\begin{aligned}&\mathbb{E}\left[\textstyle\sum_{k=0}^{n-1}\mathbb{E}\left[\,|\Delta X_{k+1} - \Delta Y_{k+1}|^2 \mid \mathcal{F}_k\right]^{\frac{1}{2}}\right] \\ &\quad\leq \sqrt{n}\mathbb{E}[\sqrt{\textstyle\sum_{k=0}^{n-1}\mathbb{E}\left[\left|\Delta X_{k+1} - \Delta Y_{k+1}\right|^2 \mid \mathcal{F}_k\right]}] \\ &\quad\leq \sqrt{n}\sqrt{\textstyle\sum_{k=0}^{n-1}\mathbb{E}\left[\left|\Delta X_{k+1} - \Delta Y_{k+1}\right|^2\right]} \\ &\quad= \sqrt{n}\sqrt{\mathbb{E}[|X_n - Y_n|^2]}\end{aligned}$$

$\square$

CHAPITRE 3

Applications aux jeux d'échange financiers et aux modèles de prix.

On s'intéresse dans ce chapitre à l'approche stratégique des modèles de prix sur des marchés avec asymétrie d'information introduite dans De Meyer [**25**]. On présente un modèle général de jeu répété d'échange à somme nulle entre deux agents risque-neutres asymétriquement informés. Les agents peuvent échanger plusieurs actifs, et on s'intéresse particulièrement au cas où il y a un actif sous-jacent et plusieurs produits dérivés. La fluctuation des prix dans ce modèle est endogène et résulte de l'asymétrie d'information et du comportement stratégique des agents. On montre que la valeur de ces jeux en temps discret ainsi que les processus de prix à l'équilibre convergent respectivement vers la valeur et les solutions d'un problème limite en temps continu quand le nombre de période d'échange tend vers l'infini. De plus, ce problème limite est essentiellement indépendant du mécanisme d'échange du jeu.

Dans le cas d'un modèle à un actif, il a été prouvé dans [**25**] que le processus de prix limite est unique et appartient à la classe appelée CMMV. On prouve que ce résultat est robuste face à l'introduction dans le modèle de plusieurs produits dérivés monotones. Néanmoins, on prouve que ce résultat n'est plus vrai dans des cas particuliers non-monotones, et on propose un exemple où le processus de prix limite n'est plus unique.

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1. Introduction

Many financial models in use on the stock markets rely on the definition of a class of possible price dynamics for an underlying asset. By price dynamic, we mean a probability distribution describing the future random evolution of the price of a risky asset in continuous time. Once this class is defined, statistical calibration tools as well as numerical methods to compute derivatives' prices and hedging strategies are developed and their formulation strongly depend on the considered class. This dependence outlines the importance of the problem consisting in the choice of a particular class.

This work is following the approach to this model selection problem initiated in the work of [**31**]. This approach is different from classical financial models since it relies on a strategic analysis using game theoretical tools, where the agents are not considered as price-taker. In this setting, the fluctuations of the prices result from an asymmetry of information and from agents' strategic behavior. The randomness is therefore endogenous and this differs from the usual justifications. Indeed, in many classes of price dynamics (as Bachelier, Black and Scholes, and more generally the class of diffusion processes) the randomness appearing in the stock prices is modeled by a Brownian motion and the usual justification for its appearance is the following: prices depend on a long list of external random parameters whose effects aggregate in a Brownian motion due to an implicit central limit theorem.

The model introduced in [**31**] consists in a repeated exchange game between two risk-neutral asymmetrically informed players. The first player is an informed agent, that has to be thought as an institutional investor having a better access to information, this advantage being known publicly. The second player is an uninformed agent, who just knows that his opponent is informed. Therefore, each move of the first player on the market is analyzed by the other to infer its informational content. A naive use of information would be completely revealing, and lead to a loss of this strategic advantage for the next periods of trade. The informed agent has thus to care about how much information his actions will reveal. His optimal behavior in this game is to select his actions using some random lotteries, misleading this way the beliefs of the uninformed player. These random noises introduced at each period in the repeated trade will aggregate in a Brownian motion. Aggregation has to be understood as the convergence of a sequence dicrete-time approximations. Indeed, equilibrium strategies in this repeated game define a random price process. This process is a discrete-time process, but any sequence of these processes converges to a precise price dynamic (continuous-time) when the time between two trading periods tends to zero.

More recently, this convergence result has been generalized in [**25**]. The construction and the arguments developed in [**31**] are extended to a large class of repeated exchange games. The main result is that, independently of the exchange mechanism of the game, the same asymptotic price dynamics are obtained. The class of these dynamics, called continuous martingales of maximal variation (CMMV), is a subclass of the local volatility diffusion processes that contains Bachelier's [**4**] and Black and Scholes' [**11**] dynamics as particular cases. Due to this invariance result, the class CMMV appears as quite robust, and a natural question to ask is whether this result still holds in more general settings (non-zero sum games, risk aversion, multi-assets game

...). In this paper, we aim to analyze a multi-asset model where the informed agent is dealing various assets at a time. The main motivation for this extension is the following.

The robustness' result proved in [**25**] suggests that the class CMMV could be used in financial models as the class of possible dynamics for the stock prices, since it provides an economic justification for this class. However, in this model, agents are trading only one asset, and a price dynamic is mainly a tool to obtain pricing formulae for derivatives. The main question we ask is then : Is this result still robust if we modify the model so that the agent can also trade derivatives? More precisely, assuming that we obtain a multi-dimensional distribution for the prices of both the underlying asset and the derivatives introduced in the model, is the marginal distribution of the underlying asset still the same, and in particular still in the class CMMV ?

Our results rely on the following mathematical result, generalizing the convergence result in [**25**] for the unidimensional case. This result characterizes the asymptotic behavior of martingales of maximal M-variation. Given a real-valued function M defined on the set of probabilities on $\mathbb{R}$, the M-variation of a discrete-time stochastic process $(X_1, .., X_n)$ is the following functional :

$$\mathcal{V}_n^M(X) = \mathbb{E}[\sum_{k=1}^{n} M([(X_k - X_{k-1} \mid X_1, .., X_{k-1}])]$$

where $[(X_k - X_{k-1}) \mid X_1, .., X_{k-1}]$ is the conditional distribution of the increment $(X_k - X_{k-1})$ of the process given the past values $(X_1, .., X_{k-1})$. A multi-dimensional extension of this theorem is proved in chapter 2 for general functions M, including the case of the value function of an exchange game with asymmetric information.

The main difficulty appearing in the multi-asset models that prevent us from obtaining a straightforward generalization of the convergence result given in [**25**] is an effect that we will call increasing information due to the introduction of a new asset in the model. Suppose that the two players are exchanging two risky assets A and B, B being a derivative on A. The first player is assumed to have some information concerning the liquidation value of A at some future date. He will therefore try to take benefit of this advantage on both markets, using his information about A to trade B. At this point, it is quite natural to ask if the situation can be reduced to two separate games. It would be possible if the informed agent was not known to be informed and acting on both markets. He could in this case use optimal strategies developed in the one-asset models, using all information available for each asset independently. In our case however, the decisions of the uninformed player are made by gathering observations coming from both markets. Since observations made on market B contain indirectly information on A (and reciprocally), the second player is a priori more informed than if the markets were completely separated. This increasing information effect can dramatically change the type of price dynamics.

In this work, we introduce a general two-assets model, including as a particular case the couple formed by an asset and some derivative on it, but allowing for more complicate, possibly probabilistic, dependencies. All the results obtained extend directly to a model with a finite number of assets. We define a repeated exchange game between two risk-neutral players in which one of them is assumed to be informed about the liquidation value of the two risky assets. In this general setting, the different assets cannot be separated, and the results of [**25**] do not apply due to the above mentioned effect of increasing information.

The organization is as follows: The next section is a reminder of notations and results of [**25**]. From section 3 to 8, we study the structure of a particular two assets game which is an extension of the game introduced in [**31**]. Section 3 presents the model as well as the strategies and the payoff function in that game. In section 5, we relate the one-shot two-assets model with two independent one-asset models. We characterize the price process at equilibrium in the game of length n as a maximizer of an optimization problem for martingales in section 6. We discuss the asymptotic behavior of this game as an approximation of a continuous time optimization problem in section 8 in order to give some insight of the main results of chapter 2.

In section 1.2, we show how to deduce from the results of chapter 2 the asymptotic behavior of the value of our games.

Using this result in section 9, we study the special case in which the second asset B is a monotonic derivative on A. The main result (Theorem 9.2) in this section says that the dynamic of A is not perturbed by the introduction of B in the model. It shows that the effect of increasing information does not modifies the prices when we introduce in the one asset-model derivatives with the same maturity date whose prices are monotonic functions of the underlying asset's price at that date, typically european options. This gives a positive answer to the above mentioned question, and states an additional robustness result for the class CMMV.

The last section (10) treats the case of a general link between A and B including non monotonic or probabilistic dependencies. The problem of characterizing the price dynamics is formulated as a dual stochastic control problem in continuous time. We provide an explicit solution of this problem for a particular case of non monotonic dependency between A and B liquidation values, using a dual PDE formulation. In contrast to Theorem 9.2, this solution provides an example where the answer is negative: the price dynamic of the underlying asset is modified by the introduction of the derivative and is no more in the class CMMV due to the effect of increasing information. We also give some examples of more general probabilistic dependencies where there is non-uniqueness of the price dynamic.

2. The one asset model

This paper is concerned with a multi-assets model that generalizes the one asset case analyzed in [**25**]. It will be useful to remind both results and notations of that paper. The following description is only formal and we refer to [**25**] for the precise statements.

The model consists in a zero-sum game denoted by $G_n(\mu^A)$, in which two agents (Player 1 and 2) are trading a risky asset A against a numéraire N.

Information structure. At the beginning of the game, which will correspond for us to the time $t = 0$, Player 1 receives some private message concerning the risky asset A. At some fixed future date, for us $t = 1$, P1's message will be publicly disclosed. The price L^A of asset A at time $t = 1$ is called the liquidation value of A. It will depend on P1's message. Since L^A is the only useful content of P1's information in this model, we may assume that nature initially chooses L^A with a probability μ^A over $\mathbb{R}$, informs P1 but not P2 who only knows μ^A. The liquidation value of N will be for simplicity fixed to 1.

The repeated exchange game. Before the disclosure date $t = 1$, players exchange (trade) repeatedly asset A and N during n consecutive rounds. Each round is described by the same general exchange mechanism, which is defined as a zero-sum game (I, J, T), where I, J are the respective action sets of P1 and P2 and where T is a transfer function from $I \times J$ to $\mathbb{R}^2$. If players' actions are (i, j), then $T(i, j) = (R(i, j), N(i, j))$ where $R(i, j)$ and $N(i, j)$ represent respectively the number of A and N shares received by P1.

At round k ($k = 1, ..., n$), the players have to select a pair of actions (i_k, j_k) independently of each others, based on their past observations and private information. Next the resulting transaction takes place and the actions are announced publicly. This transaction is summarized by the following relations. Let $y_k = (y_k^A, y_k^N)$ and $z_k = (z_k^A, z_k^N)$ denote P1's and P2's portfolios after round k, then

$$y_k = y_{k-1} + T(i_k, j_k) \quad ; \quad z_k = z_{k-1} - T(i_k, j_k)$$

The payoff function. Both players are assumed to have sufficiently large initial endowments, and therefore the constraints $y_k \geq 0$ and $z_k \geq 0$ are ignored in this model. The players are supposed to be risk neutral so that the utility they aim to maximize is the expected value of their final portfolio. P1's utility is then : $\mathbb{E}[y_n^A L^A + y_n^N]$. Since y_0 is initially fixed, its liquidation value is independent of players' moves. It can thus be subtracted from P1's utility without affecting his behavior in the game. The same argument can be applied to P2 and this amounts to assume $y_0 = z_0 = (0, 0)$.

Players are assumed to use behavioral strategies: at each round, they can use a lottery depending on their past observations and private information to select their actions (see chapter 1 for a formal definition).

The price process. In this abstract model, the price L_k^A of asset A at stage k is defined as the expected liquidation value given P2's information: $L_k^A := \mathbb{E}[L^A \mid i_1, .., i_k, j_1, ..j_k]$ (it is the price at which P2 would agree to trade with another risk-neutral player having the same information).

Asymptotic results. The main result in [**25**] concerns the asymptotic of the price process at equilibrium $(L_k^{A,n})_{k=0,...,n}$, as the number of rounds goes to ∞. Since the players use a lottery to select their actions, at equilibrium in $G^n(\mu^A)$, the sequence of prices $(L_k^{A,n})_{k=1,..,n}$ is a stochastic process. The n transaction rounds occur between the date $t = 0$ when P1 receives the message and the date of public disclosure $t = 1$. Assuming that round k occurs at time $t = \frac{k}{n}$ we can then extend this price process in a continuous time process Π_t^n, piecewise constant on the intervals $[\frac{k}{n}, \frac{k+1}{n})$. Precisely, with $\lfloor a \rfloor$ the greater integer less or equal to a, and $L_0^{A,n} = \mathbb{E}(L^A)$, we define:

$$\forall t \in [0, 1] \quad \Pi_t^n = L_{\lfloor nt \rfloor}^{A,n}$$

When n becomes large, the time between two transactions goes to zero, and the price process appears naturally as an approximation of a continuous-time price process. The limit, if it exists, of the processes Π_t^n, represents then a continuous-time "equilibrium" price process. The main result in [**25**] is that, under very weak and natural hypotheses on the exchange mechanism described below, Π^n converges in distribution [1] to the continuous martingale Π^{μ^A} of maximal variation (CMMV). This martingale Π^{μ^A} is defined as : $\Pi^{\mu^A}_t := \mathbb{E}[f_{\mu^A}(B_1) \mid \mathcal{G}_t]$ where B is a Brownian motion on its natural filtration $\mathcal{G}$ and f_{μ^A} the unique right-continuous nondecreasing function such that $f_{\mu^A}(B_1)$ is μ^A-distributed.

The class of processes Π^{μ^A} is therefore quite universal or robust, in the sense that it is independent of the "natural" exchange mechanism used by the players. An exchange mechanism is said to be natural if it satisfies the following five hypotheses for μ^A in some fixed subset of probabilities :

H1) *Existence of a value*: For all n and μ^A the game $G_n(\mu^A)$ has a value $v_n(\mu^A)$ and both players have optimal strategies.

H2) *Continuity* : The value v_1 is K-Lipschitz with respect to the Wasserstein distance of order p for a $p \in [1, 2)$. (see appendix for a precise definition of this distance)

H3) *Invariance with respect to the numéraire scale* that can be expressed in terms of v_1: For all random variable X we denote $[X]$ the law of X. With that notation, the hypothesis becomes for all $\alpha \geq 0$, for all X : $v_1([\alpha X]) = \alpha v_1([X])$

H4) *Invariance with respect to the risk-less part of the risky asset*:
For all $\beta \in \mathbb{R}$, for all X, $v_1([X + \beta]) = v_1([X])$.

H5) *Positive value of information*:
v_1 is nonnegative and there exists μ^A such that $v_1(\mu^A) > 0$.

An example of Natural Mechanism. The following mechanism was introduced in [**31**] where the associated game $G_n(\mu^A)$ was studied for distributions concentrated on $\{0, 1\}$. The generalization for any distribution was made in [**30**]. The convergence result was first proved using this mechanism, and it provides a concrete example that fulfills all the hypotheses needed to apply the general result obtained in [**25**].

This example is a simplified zero-sum version of bid-ask competition between market makers. More precisely, for each round of transaction, both players have to post a price for 1 unit of the risky asset. Then, the maximal bid wins and one share is exchanged at this price (directly between the players). If both bids are equal, no transaction happens. Let p denote the price posted by P1 (respectively q for P2), and (I, J, T) this mechanism, then we have:

$$I = J = \mathbb{R}, \quad and \quad T(p,q) = (R(p,q), N(p,q)) \quad with :$$

$$R(p,q) = \mathbb{1}_{(p>q)} - \mathbb{1}_{(q>p)} \qquad N(p,q) = \mathbb{1}_{(q>p)}q - \mathbb{1}_{(p>q)}p \tag{2.1}$$

where $\mathbb{1}_{(p>q)}$ is the function equal to 1 if $p > q$ and to 0 otherwise. Other examples are discussed in [**25**], and all our results in the multi-asset setting extend to any natural mechanism except the existence of the value and of optimal strategies which is specific to the payoff function of the game unless strong regularity assumptions are made on T, I, J.

1. Convergence in distribution has to be understood as convergence of probability measures on the space $\mathbb{D}([0,1], \mathbb{R}^p)$ of rcll functions endowed with the Skorokhod topology. The result in [**25**] is stated using convergence in finite-dimensional distributions. But using the result of Aldous [**1**], this convergence implies the stronger convergence we mention here since the limit process has continuous trajectories.

3. An exchange game model with two assets

The two-assets game Γ_n. Let us now describe the two-assets game using the particular exchange mechanism described in the previous section. At the beginning of the game (date $t = 0$), the pair of liquidation values $L = (L^A, L^B)$ is chosen at random according to a probability distribution μ over $\mathbb{R}^2$, and P1 is informed of L. P2 knows only μ and that P1 has been informed of L. Transactions take place during n consecutive rounds up to date 1, and the **same** exchange mechanism is used on both markets. It means that both players, at each round and for each asset, have to post a price. Let $p_k = (p_k^A, p_k^B)$ denote the prices posted by P1 at round k, (respectively $q_k = (q_k^A, q_k^B)$ for P2). If $y_k = (y_k^A, y_k^B, y_k^N)$ denotes player 1's portfolio after round k

$$y_k = y_{k-1} + t(p_k, q_k) \qquad with:$$

$$t(p,q) = \Big(R(p^A, q^A)\ ,\ R(p^B, q^B)\ ,\ N(p^A, q^A) + N(p^B, q^B)\Big)$$

where R and N are given by (2.1). Players are risk-neutral and aim to maximize the expected value of their final portfolios.

We denote $\Gamma_n(\mu)$ the two assets game, with μ the distribution of the vector of liquidation values $L = (L^A, L^B)$. The one asset-game using the same mechanism will be denoted as in the previous section $G_n(\mu^A)$ where μ^A is the distribution of the liquidation value L^A. For a given bivariate distribution μ, the pair of univariate marginal distributions of μ will be denoted by (μ^A, μ^B). The study the possible effect of the introduction of a second asset in the model on the prices reduces in this setting to the comparison of the equilibrium price processes for the asset A in $G_n(\mu^A)$ and $\Gamma_n(\mu)$. Note that for the case in which B is a derivative on A, the information structure is the same in $G_n(\mu^A)$ and $\Gamma_n(\mu)$. Indeed, since L^B is a function of L^A, knowing (L^A, L^B) is equivalent to knowing L^A. The comparison between these two games studies then precisely the effect of allowing the players to trade derivatives in this model.

Strategies in $\Gamma_n(\mu)$. We assume that $\mu \in \Delta^1(\mathbb{R}^2)$ where $\Delta^p(\mathbb{R}^2)$ denotes the set of Borelian probabilities on $\mathbb{R}^2$ with finite moment of order p.

At round k, the choice made by P2 is based on his past information

$$(p_i, q_i)_{i \le k-1} \in (\mathbb{R}^2)^{2(k-1)}.$$

A behavioral strategy τ for P2 in $\Gamma_n(\mu)$ is then a sequence $(\tau_1, ..., \tau_n)$ of transition probabilities

$$\tau_k : (\mathbb{R}^2)^{2(k-1)} \to \Delta(\mathbb{R}^2).$$

Let $\mathcal{T}_n$ denote the set of these strategies. P1 can also use his private information to make his choice, so the prices he post at round k are depending on

$$(L, (p_i, q_i)_{i \le k-1}) \in \mathbb{R}^2 \times (\mathbb{R}^2)^{2(k-1)}.$$

A behavioral strategy for P1 is then a sequence $\sigma = (\sigma_1, ..., \sigma_n)$ of transition probabilities

$$\sigma_k : \mathbb{R}^2 \times (\mathbb{R}^2)^{2(k-1)} \to \Delta(\mathbb{R}^2).$$

The triplet (μ,σ,τ) induces a unique probability $\Pi_{(\mu,\sigma,\tau)}$ on $(\mathbb{R}^2)\times(\mathbb{R}^2)^{2n}$. Formally, the payoff function is then

$$\begin{aligned} g_n(\mu,\sigma,\tau) = E_{\Pi_{(\mu,\sigma,\tau)}}[L^A \sum_{k=1}^{n}(R(p_k^A,q_k^A)) + \sum_{k=1}^{n}(N(p_k^A,q_k^A)) \\ +L^B \sum_{k=1}^{n}(R(p_k^B,q_k^B)) + \sum_{k=1}^{n}(N(p_k^B,q_k^B))] \end{aligned}$$

The quantity above is not always well defined for integrability reasons, so we have to restrict the sets of strategies. As $\mu \in \Delta^1(\mathbb{R}^2)$ the quantities:

$$L^i(R(p_k^i,q_k^i)) = L^i(\mathbb{1}_{p_k^i>q_k^i} - \mathbb{1}_{q_k^i>p_k^i}) \quad \text{for } i\in\{A,B\}$$

are always integrable, so we will concentrate on the remaining part of the payoff . A strategy σ for P1 is admissible if:

$$\forall \tau \in \mathcal{T}_n, \forall k=1,..,n, \forall i \in \{A,B\}, \quad \mathbb{E}_{\Pi_{(\mu,\sigma,\tau)}}\left[\left(N(p_k^i,q_k^i)\right)^-\right] < \infty$$

where for a real number x, x^- denotes the negative part of x. $\Sigma_n(\mu)$ will be the set of admissible strategies for P1 in $\Gamma_n(\mu)$. Restricting the set of P1's strategies to admissible ones, the payoff is always well defined in $\mathbb{R}\cup\{+\infty\}$.

Equilibrium. Let us recall the following notions of zero-sum games theory. We say that the strategy σ of P1 can guarantee C in $\Gamma_n(\mu)$ if

$$\underset{\tau\in\mathcal{T}_n}{inf}\, g_n(\mu,\sigma,\tau) \geq C$$

Similarly, a strategy τ can guarantee C is

$$\underset{\sigma\in\Sigma_n(\mu)}{sup}\, g_n(\mu,\sigma,\tau) \leq C$$

The maximal amount P1 can guarantee is therefore

$$\underline{V}_n(\mu) = \underset{\sigma\in\Sigma_n(\mu)}{sup}\ \underset{\tau\in\mathcal{T}_n}{inf}\, g_n(\mu,\sigma,\tau)$$

A strategy σ is said to be optimal optimal if it guarantees $\underline{V}_n(\mu)$. The minimal amount P2 can guarantee is:

$$\overline{V}_n(\mu) = \underset{\tau\in\mathcal{T}_n}{inf}\ \underset{\sigma\in\Sigma_n(\mu)}{sup}\, g_n(\mu,\sigma,\tau)$$

A strategy τ is said to be optimal if it guarantees $\overline{V}_n(\mu)$. We always have $\underline{V}_n(\mu)\leq\overline{V}_n(\mu)$ and when equality holds, the game is said to have a value

$$V_n(\mu) \triangleq \underline{V}_n(\mu) = \overline{V}_n(\mu).$$

An equilibrium of the game is a pair of optimal strategies (σ,τ).

Reduced strategies in $\Gamma_n(\mu)$. We introduce the notion of reduced strategies, which are simply strategies that do not depend on P2's past actions.
A reduced strategy τ for P2 is a sequence $(\tau_1, ..., \tau_n)$ of transition probabilities

$$\tau_k : (\mathbb{R})^{k-1} \to \Delta(\mathbb{R}).$$

A reduced strategy for P1 is a sequence $\sigma = (\sigma_1, ..., \sigma_n)$ of transition probabilities

$$\sigma_k : \mathbb{R} \times (\mathbb{R})^{k-1} \to \Delta(\mathbb{R}).$$

Given a reduced strategy σ of P1, the pair (μ, σ) induces a probability on $\mathbb{R}^2 \times (\mathbb{R}^2)^n$. Therefore, if

$$\mathbb{E}_{(\mu,\sigma)}[\sum_{k=1}^{n} \mid p_k^A \mid + \mid p_k^B \mid] < \infty$$

then σ is admissible. Indeed, this follows from the following obvious inequality

$$N(p_k^i, q_k^i) = \mathbb{1}_{(q_k^i > p_k^i)} q_k^i - \mathbb{1}_{(p_k^i > q_k^i)} p_k^i \geq -|p_k^i|$$

Remark 3.1: *Since our results rely on the recursive structure of this repeated game, we need the following admissibility criterion. If there exists $M > 0$ such that with probability 1*

$$\forall k = 1, .., n,\ \forall i = A, B, \quad \mathbb{E}_{(\mu,\sigma)}[|p_k^i||p_1, .., p_{k-1}] \leq M\mathbb{E}_{(\mu,\sigma)}[|L^i||p_1, .., p_{k-1}]$$

then σ is admissible.

The main property of the game in reduced strategies is that the action of P2 at step k does not influence the payoff of next stages. Therefore, to play a best reply against some reduced strategy σ of P1, P2 has just to maximize his stage payoff given past actions of P1. In other words, P2 can be seen as a succession of players facing P1, being informed only of the past actions of P1. The next proposition shows that an optimal strategy in the game restricted to reduced strategies is still optimal in the initial game.

Proposition 3.1: *If a strategy guarantees the quantity C in the game $\Gamma_n(\mu)$ where the strategy sets of the players are restricted to reduced (admissible) strategies, then it guarantees C in the initial game.*

Proof. see proposition 1.4 in chapter 1. □

4. The one asset game G_n

The set of admissible strategies in G_n are defined in the same way as above (with an obvious truncation). The following result was proved in [**30**]:

Theorem 4.1: *For $\mu^A \in \Delta^1(\mathbb{R})$, the game $G_n(\mu^A)$ has a value $v_n(\mu^A)$ and both players have an optimal strategy. Moreover, the function v_1 is given by:*

$$v_1(\mu^A) = \int_0^1 F_{\mu^A}^{-1}(u)(2u - 1)du \tag{4.1}$$

where for any probability ν over $\mathbb{R}$, F_ν denotes the distribution function and F_ν^{-1} its generalized right-continuous inverse.

We introduce now some notations and properties for set of joint probabilities with constraint on the marginals.

NOTATION 10: *Let X,Y be two polish spaces and Q,R two subsets of the set of Borelian probabilities $\Delta(X)$ and $\Delta(Y)$. Then $\mathcal{P}(Q,R)$ denotes the set of probabilities over $X\times Y$ whose marginal distributions over X and Y belongs respectively to Q and R. If $Q=\{\mu\}$, we will simply write $\mathcal{P}(\mu,R)$. Unless otherwise stated, topological properties of such sets refer to the usual weak topology.*

LEMMA 4.1: *Let X,Y be two polish spaces and Q,R two tight (resp. closed, convex) subsets of $\Delta(X)$ and $\Delta(Y)$. Then the set $\mathcal{P}(Q,R)$ is itself tight (resp. closed, convex).*

Let us now recall the following well-known result concerning maximal covariance functions on the line, which is a particular case of the Monge-Kantorovitch optimal transportation problem.

LEMMA 4.2: *If $\mu\in\Delta^p(\mathbb{R})$ with $p\geq 1$ and $\nu\in\Delta^q(\mathbb{R})$ with $\frac{1}{p}+\frac{1}{q}=1$, then:*

$$max\{\mathbb{E}[XY]\mid X\sim\mu, Y\sim\nu\}=\int_0^1 F_\mu^{-1}(u)F_\nu^{-1}(u)du$$

If F_ν is bijective, the law of $(F_\mu^{-1}(F_\nu(Y)),Y)$ induced by some random variable $Y\sim\nu$ is an optimal solution, and the function $f=F_\mu^{-1}\circ F_\nu$ is the unique nondecreasing function (up to equality ν a.e.) such that $f(Y)\sim\mu$. Moreover if $X\sim\mu$ is a r.v. defined on the same space as $Y\sim\nu$ and such that (X,Y) is an optimal solution, then $X=(F_\mu^{-1}(F_\nu(Y))$ a.s..

PROOF. see theorem 6.0.2 in [**2**]. □

REMARK 4.1: *Any pair of random variable optimal in the preceding lemma is called comonotonic. Although the same problem does not have explicit solutions in higher dimensions, this notion of comonotonicity allows to solve it in some particular cases. The symmetric notion of an antimonotonic pair is a pair (X,Y) such that there exists some variable U and $X=f(U)$, $Y=g(U)$, with f nondecreasing and g nonincreasing.*

We deduce directly from this lemma the following representation formula for v_1

$$v_1(\mu^A)=max\{\mathbb{E}[L^AU^A]\mid L^A\sim\mu^A\,,\,U^A\sim\mathcal{U}_{[-1,1]}\} \tag{4.2}$$

$$=\max_{\pi\in\mathcal{P}(\mu,\mathcal{U}_{[-1,1]})}\int xyd\pi(x,y) \tag{4.3}$$

where $\mathcal{U}_{[-1,1]}$ denotes the uniform distribution on $[-1,1]$ and where the second line is just an integral reformulation of the first one. Let us now mention some useful properties of v_1.

LEMMA 4.3: *v_1 fulfills the properties $H2$ to $H5$ with $p=K=1$ on the set $\Delta^1(\mathbb{R})$, together with the following symmetry hypothesis*

$$\forall\alpha\in\mathbb{R},\ v_1[\alpha Y]=\mid\alpha\mid v_1[Y]$$

PROOF. Let $\nu_1,\nu_2\in\Delta^1(\mathbb{R})$ and X_1,X_2 random variables such that $X_i\sim\nu_i$ and $\mathbb{E}[|X_1-X_2|]=d_{W_1}(\nu_1,\nu_2)$. Then for any uniform random variable U on $[-1,1]$ defined on the same probability space, we have

$$|\mathbb{E}[X_1U]-\mathbb{E}[X_2U]|\leq d_{W_1}(\nu_1,\nu_2)$$

and property $H2$ follows easily. properties $H3$ to $H5$ and the symmetry hypothesis are obvious consequences of the linearity of the expectation and of the fact that the uniform law is centered and symmetric. □

We recall finally two technical lemmas proved in [**31**] [2], that we will use to construct optimal strategies in the one asset game G_1, and that we will use to construct optimal strategies in Γ_n.

LEMMA 4.4: *For $\mu^A \in \Delta^1(\mathbb{R})$, define $f(u) = \frac{1}{u^2}\int_0^u 2sF_{\mu^A}^{-1}(s)ds$. Then:*

$$\begin{aligned}\min_{q\in\mathbb{R}} \int_0^1 &\Big(F_{\mu^A}^{-1}(u)(\mathbb{1}_{f(u)>q} - \mathbb{1}_{q>f(u)}) + \mathbb{1}_{q>f(u)}q - \mathbb{1}_{f(u)>q}f(u)\Big)du \\ &= \int_0^1 (2s-1)F_{\mu^A}^{-1}(s)ds\end{aligned}$$

and the minimum is reached for any $q \in f(]0,1[)$. Moreover, $f \le F_{\mu^A}^{-1}$ and

$$\int_0^1 F_{\mu^A}^{-1}(u) - f(u)du = \int_0^1 (2u-1)F_{\mu^A}^{-1}(u)du \le \int_0^1 \left|F_{\mu^A}^{-1}(u) - m(\mu^A)\right| du$$

with $m(\mu^A) = \int_0^1 F_{\mu^A}^{-1}(u)du$.

LEMMA 4.5: *Let h be a continuous convex function defined on $[-1,1]$. Define:*

$$g(u) = \frac{1}{u^2}\int_0^u 2sh'(2s-1)ds \quad \text{where } h' \text{ is the right-hand derivative of } h$$

We have:

$$\begin{aligned}\max_{p\in\mathbb{R}} &\left[h\left(\int_0^1 (\mathbb{1}_{p>g(u)} - \mathbb{1}_{g(u)>p})du\right) + \int_0^1 (\mathbb{1}_{g(u)>p}g(u) - \mathbb{1}_{p>g(u)}p)du\right] \\ &= \int_0^1 h(2s-1)ds\end{aligned}$$

5. The one shot game $\Gamma_1(\mu)$

In $\Gamma_1(\mu)$, since there is only one transaction round, P1 has not to care about the information his action will reveal. Therefore, playing that game is equivalent to playing independently the two games $G_1(\mu^A)$ and $G_1(\mu^B)$. Therefore, both players can play optimally in both markets using any coupling of optimal strategies in $G_1(\mu^A)$ and $G_1(\mu^B)$. We deduce from this observation the following result

PROPOSITION 5.1: *For all $\mu \in \Delta^1(\mathbb{R}^2)$,$\Gamma_1(\mu)$ has a value*

$$V_1(\mu) = v_1(\mu^A) + v_1(\mu^B)$$

and both players have optimal strategies.

PROOF. We will only explain briefly how to formalize the above argument in order to obtain a rigorous proof. The payoff function has been defined in order to ensure that the payoff is

2. resp. lemma 5.1 p.15 and 7.1 p.22. See also the construction of optimal strategies in [**31**], the proof being the same.

always the sum of two expectations, resulting respectively from the exchanges on market A and B. The "A" expected payoff is :

$$E_{\Pi_{(\mu,\sigma,\tau)}}[L^A R(p^A, q^A)) + N(p^A, q^A)]$$

and the relevant part of players strategies to compute it reduces to a pair of one-asset game's strategies, precisely : the conditional distribution of p^A given L^A and the marginal distribution of q^A, which define respectively a strategy in $G_n(\mu^A)$ for P1 and P2. Applying the same decomposition for the second part of the payoff, the relevant part of any strategy in $\Gamma_1(\mu)$ can be represented by a pair of strategies in $G_n(\mu^A)$,$G_n(\mu^B)$. The reverse way, coupling two G_n strategies into a Γ_n strategy, can be done for example this way : given (σ^A, σ^B) strategies in the games $(G_1(\mu^A), G_1(\mu^B))$ define a strategy $(\widehat{\sigma^A, \sigma^B})$ in $\Gamma_1(\mu)$ by:

$$(\widehat{\sigma^A, \sigma^B})(L^A, L^B) = \sigma^A(L^A) \otimes \sigma^B(L^B)$$

and similarly $(\widehat{\tau^A, \tau^B}) = \tau^A \otimes \tau^B$. Since admissibility as well as measurability is clearly preserved in these operations, the equivalence mentioned above is straightforward to prove. □

We need for the sequel to prove a slightly more precise result.

DEFINITION 5.1: *An optimal admissible selection for P1 is a transition probability, that is a measurable application :*

$$\sigma : \Delta^1(\mathbb{R}^2) \times \mathbb{R}^2 \to \Delta(\mathbb{R}^2)$$

such that $\forall \mu \in \Delta^1(\mathbb{R}^2),\ \sigma(\mu, \cdot) \in \Sigma^1(\mu)$ *and:*

$$\forall \mu, \forall \tau \in \mathcal{T}^1, \quad g_1(\mu, \sigma(\mu,\cdot), \tau) \geq V_1(\mu)$$

Moreover, we require for admissibility that there exists $M > 0$ *such that*

$$\forall i = A, B, \quad \mathbb{E}_{(\mu,\sigma(\mu,\cdot))}[|p_1^i|] \leq M \mathbb{E}_\mu[|L^i|]$$

PROPOSITION 5.2: *P1 has an optimal admissible selection in* $\Gamma_1(\mu)$ *and:*

$$V_1(\mu) = \max\left\{ \int \langle x, y\rangle d\pi(x,y) \middle| \pi \in \mathcal{P}(\mu, H_1)\right\}$$

where $\mathcal{P}(\mu, H_1)$ *refers to bivariate marginals and* $H_1 = \mathcal{P}(\mathcal{U}_{[-1;1]}, \mathcal{U}_{[-1;1]})$ *refers to univariate marginals.*

We will need the following classical lemma for the proof, whose general form is known as the regression representation lemma.

LEMMA 5.1: *Let* X *be a random variable of distribution* ν *and* V *another random variable of distribution* $\mathcal{U}_{[0,1]}$ *independent of* X*. Define:*

$$\widetilde{F}_\nu(x) = \mathbb{P}(X < x) \quad , \quad p_\nu(x) = \mathbb{P}(X = x)$$

$$\textit{and} \quad \theta(\nu, x, \lambda) = \widetilde{F}_\nu(x) + \lambda p_\nu(x)$$

Then:

$$\theta(\nu, X, V) \sim \mathcal{U}_{[0,1]} \quad \textit{and} \quad F_\nu^{-1}(\theta(\nu, X, V)) = X \ \textit{a.s.}$$

PROOF. See theorem 2.1 in [**52**]. □

PROOF OF PROPOSITION 5.2. Given two independent random variables (V^A, V^B) with distribution $\mathcal{U}_{[0;1]}$ and independent of L, define for $i = A; B$:

$$U^i = \theta(\mu^i, L^i, V^i) \quad \Rightarrow L^i = F^{-1}_{\mu^i}(U^i) \text{ and } U^i \sim \mathcal{U}_{[0;1]}$$

If P1 plays $p^i = h^i(U^i)$, with : $h^i(\alpha) = \frac{1}{\alpha^2}\int_0^\alpha 2sF^{-1}_{\mu^i}(s)ds$, it defines a strategy σ, and using lemma 4.4, P1 guarantees with σ :

$$\begin{aligned} v_1(\mu^A) + v_1(\mu^B) &= \mathbb{E}[F^{-1}_{\mu^A}(U^A)(2U^A - 1)] + \mathbb{E}[F^{-1}_{\mu^B}(U^B)(2U^B - 1)] \\ &= \mathbb{E}[L^A(2U^A - 1) + L^B(2U^B - 1)] \end{aligned}$$

Using the above construction, the variables appearing in the last equality are defined on the same probability space and fulfill the requested marginal constraints. Therefore :

$$v_1(\mu^A) + v_1(\mu^B) \leq \max\left\{ \int \langle x, y\rangle d\pi(x,y) \middle| \pi \in \mathcal{P}(\mu, H_1)\right\}$$

The reverse inequality is clear from the definition of v_1. The optimal strategy of P1 we have constructed is a coupling (as defined in the proof of proposition 5.1) of some one-asset game strategies (σ^A, σ^B). Each one of these strategies can be summarized by its inverse distribution function which depends on (L^i, μ^i) for $i = A, B$

$$F^{-1}_{\sigma^i(L^i,\mu^i)}(u) = h^i \circ \theta(\mu^i, L^i, u)$$

These functions being jointly measurable, it defines jointly measurable transitions and since coupling preserve measurability, this strategy is an optimal selection. Admissibility follows directly from lemma 4.4. Indeed, we have

$$\mathbb{E}[|p^i - L^i|] = \int_0^1 |h^i(u) - F^{-1}_{\mu^i}(u)|du \leq \mathbb{E}[|L^i - \mathbb{E}[L^i]|]$$

and this implies $\mathbb{E}[|p^i|] =\leq 3\mathbb{E}[|L^i|]$. □

6. The game $\Gamma_n(\mu)$

In our game, since player's actions are posted prices, a natural notion for the price process is the sequence of prices posted by P1. The main results in section 9 are expressed using the abstract notion of price introduced in section 2, which does not depend on the exchange mechanism. This choice has two advantages, at first this allows to extend these results to games using abstract exchange mechanisms, and secondly, the obtained price process being by construction a martingale possesses good mathematical properties. Moreover, it will be shown at the end of section 8 that this choice is not restricting for the particular game we are studying since the sequence of prices posted by P1 has the same asymptotic behavior as the martingale price process.

The martingale price process. Given (μ,σ,τ), the following process describe the evolution of the expected price of L for P2 when updating his beliefs after each round

$$L_k \triangleq \mathbb{E}_{\Pi_{(\mu,\sigma,\tau)}}[L \mid i_q, j_q, q \leq k].$$

Define $\mathcal{F}_0 = \{\emptyset, \mathbb{R}^2 \times (I^2 \times J^2)^n\}$, $\mathcal{F}_k = \sigma(i_q, j_q, q \leq k)$ for $k = 1, .., n$ and $\mathcal{F}_{n+1} = \sigma(\mathcal{F}_n, L)$. If we add to this process a final value $L_{n+1} = L$, then $(L_k, \mathcal{F}_k)_{k=0,..,n+1}$ is a $\mathbb{R}^2$-valued martingale with terminal law μ or equivalently the law of L_n is Blackwell dominated by μ (see section 12 in chapter 2). This process is very important in the analysis of the game: it indicates how much information has been revealed by player 1. If P1's move at stage k depends strongly on L, then L_k will be in some sense close to the true value L.

DEFINITION 6.1: *$\mathfrak{M}_n(\mu)$ is the collection of martingales $(X_k, \mathcal{F}_k)_{k=1,..,n}$ defined of some filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_k)_{k=1,..,n}, \mathbb{P})$, of length n and whose final distribution is Blackwell dominated by μ ($[L_n] \preceq \mu$). By convention, we set $\mathcal{F}_0 = \{\Omega, \emptyset\}$.*

By choosing a strategy, P1 is in fact choosing a rate of revelation, that is a price martingale in $\mathfrak{M}_n(\mu)$ and as argued in [**25**] the optimal price martingale will have to solve a maximization problem. In order to state formally this result, we need the following definition

DEFINITION 6.2: *We call V_1-variation of a martingale $(X, \mathcal{F}) \in \mathfrak{M}_n(\mu)$ the following functional*

$$\mathcal{V}_n(X, \mathcal{F}) = \mathbb{E}[\sum_{k=1}^{n} V_1([X_k - X_{k-1} \mid \mathcal{F}_{k-1}])]$$

The maximal V_1-variation is then the function defined for all $\mu \in \Delta^1(\mathbb{R})$ by

$$\overline{\mathcal{V}_n^{V_1}}(\mu) = \underset{(X,\mathcal{F})\in\mathfrak{M}_n(\mu)}{sup} \mathcal{V}_n^{V_1}(X, \mathcal{F}) \tag{6.1}$$

THEOREM 6.1: *For all $\mu \in \Delta^1(\mathbb{R}^2)$,*

$$\underline{V}_n(\mu) = \overline{\mathcal{V}_n^{V_1}}(\mu).$$

For all $(X, \mathcal{F}) \in \mathfrak{M}_n(\mu)$, P1 has a reduced strategy that guarantees $\mathcal{V}_n^{V_1}(X, \mathcal{F})$. If the game $\Gamma_n(\mu)$ has a value and if both players play an optimal strategy, the induced a posteriori martingale solves the problem (6.1)*.*

PROOF. At first, from property H4, the definition of maximal variation given gere coincide cith the definition of chapter 1. Then, despite the fact the the payoff function can be unbounded, the proof follows words for words from the proofs of proposition 1.2 and 1.3. Just replace the optimal strategies of P1 by the admissible measurable selection constructed in proposition 5.2 and note that the payoff is always well-defined. □

Existence of an optimal strategy for player 1 could be easily deduced from this result if we can show that there is a martingale maximizing the V_1-variation. We will derive this result from an alternative representation of $\overline{\mathcal{V}}_n$, which will be moreover useful to construct optimal strategies for P2 and to illustrate the asymptotic results in section 8.

Let us now introduce some notations.

DEFINITION 6.3:

– *The vector* $(U_k)_{k=1,\dots,n} \in (\mathbb{R}^2)^n$ *is multi-uniform (on [-1,1]) if:*

$$\forall k \in \{1,\dots,n\}, \quad [U_k \mid \mathcal{F}_{k-1}] \in H_1 \tag{6.2}$$

with $\mathcal{F}_{k-1} = \sigma(U_i)_{i=1,\dots,k-1}$ *and* $H_1 = \mathcal{P}\left(\mathcal{U}_{[-1,1]}, \mathcal{U}_{[-1,1]}\right)$.

– *If the property (6.2) holds for a larger filtration* $\tilde{\mathcal{F}}_k$*, we say that the vector is* $\tilde{\mathcal{F}}_k$ *multi-uniform. Note that it defines a sequence of* $\tilde{\mathcal{F}}_k$ *martingale increments.*

– $\mathcal{W}_n$ *denotes the set of probabilities on* $(\mathbb{R}^2)^n$ *induced by multi-uniform vectors, and* H_n *the set of probabilities on* $(\mathbb{R}^2)$ *induced by the vectors* $(\sum_{k=1}^n U_k) \in \mathbb{R}^2$*, as the joint law* $[(U_k)_{k=1,\dots,n}]$ *ranges through* $\mathcal{W}_n$.

Lemma 6.1: *Define*

$$\Phi_n(\mu) \triangleq \max\left\{\mathbb{E}[L^A \sum_{k=1}^n U_k^A + L^B \sum_{k=1}^n U_k^B] \;\middle|\; L \sim \mu, \quad [(U_k^A, U_k^B)_{k=1,\dots,n}] \in \mathcal{W}_n\right\}$$

$$= \max\left\{ \int \langle x, y\rangle d\pi \;\middle|\; \pi \in \mathcal{P}(\mu, H_n)\right\} \tag{$\mathcal{P}$}$$

Then $\overline{\mathcal{V}_n}(\mu) = \Phi_n(\mu)$.

Proof. From the previous definitions, the second line is just an integral reformulation of the first. We prove first the inequality $\leq$. For a given $(X, \mathcal{F}) \in \mathfrak{M}_n(\mu)$, let $\mathcal{G}_k$ denote the natural filtration of X. Define μ_k as the conditional law of X_k given $\mathcal{G}_{k-1}$. μ_k is then an $\mathcal{G}_{k-1}$-measurable r.v. with values in $\Delta^1(\mathbb{R}^2)$. Up to enlarging the probability space, we can assume the existence of a family $(V_k^i)_{k=1,\dots,n\ i=A,B}$ of independent r.v. of distribution $\mathcal{U}_{[0,1]}$, independent of $\mathcal{F}_{n+1}$. We define then a larger filtration $\tilde{\mathcal{G}}_k = \sigma(\mathcal{G}_k, (V_q^A, V_q^B)_{q\leq k})$ for $k = 1,..,n+1$ and $\tilde{\mathcal{G}}_0 = \mathcal{G}_0$. Since X is still a martingale with respect to $\tilde{\mathcal{G}}$ and the conditional distributions μ_k are not affected, we have

$$\mathcal{V}_n^{V_1}(X, \mathcal{F}) \leq \mathcal{V}_n^{V_1}(X, \mathcal{G}) = \mathcal{V}_n^{V_1}(X, \tilde{\mathcal{G}})$$

where the first inequality follows from Jensen's inequality (see lemma 5.1 in chapter 1). We define next, using notations of lemma 5.1:

$$(Y_k^A, Y_k^B) = (\theta(\mu_k^A, X_k^A, V_k^A), \theta(\mu_k^B, X_k^B, V_k^B)) \quad k = 1,\dots,n$$

It follows that $X_k^A = F^{-1}_{\mu_k^A}(Y_k^A)$ and $[Y_k^A \mid \tilde{\mathcal{G}}_{k-1}] = \mathcal{U}_{[0,1]}$. By proposition 5.2

$$V_1([X_k \mid \tilde{\mathcal{G}}_{k-1}]) = \mathbb{E}[X_k^A(2Y_k^A - 1) + X_k^B(2Y_k^B - 1) \mid \tilde{\mathcal{G}}_{k-1}] \tag{6.3}$$

The vector $((2Y_k^A - 1), (2Y_k^B - 1))_{k=1,..,n}$ is then $\tilde{\mathcal{G}}_k$ multi-uniform by construction, so we deduce the first inequality by summation of (6.3) over k.

For the reverse inequality, let us start with a random variable L of distribution μ and a multi-uniform vector $(U_k^A, U_k^B)_{k=1,\dots,n}$ defined on the same probability space Ω. We define the associated martingale as follows. Let $\mathcal{F}_0 = \{\varnothing, \Omega\}$, $\mathcal{F}_k = \sigma((U_i^A, U_i^B)_{i=1,\dots,k})$ for $k = 1,\dots,n$ and $\mathcal{F}_{n+1} = \sigma(\mathcal{F}_n, L)$. Then the martingale $(L_k, \mathcal{F}_k)_{k=1,\dots,n}$ defined by $L_k^i = \mathbb{E}[L^i \mid \mathcal{F}_k]$ for $i = A, B$ and $k = 1,..,n$ belongs to $\mathfrak{M}_n(\mu)$. Proposition 5.2 implies

$$\mathbb{E}[L_k^A U_k^A + L_k^B U_k^B \mid \mathcal{F}_{k-1}] \leq V_1([L_k \mid \mathcal{F}_{k-1}])$$

and we conclude by summation over k.

It remains to show that the maximum is reached in the definition of $\Phi_n(\mu)$. The definition of $\mathcal{W}_n$ can be reformulated as the set of distributions of vectors $(U_k^A, U_k^B)_{k=1,\dots,n}$ verifying :

– All one dimensional marginals distributions are $\mathcal{U}_{[-1,1]}$

– $\forall k \in \{1,\dots,n\}, \forall i \in \{A,B\}$, U_k^i is independent of $(U_m^A, U_m^B)_{m=1,\dots,k-1}$

These constraints are weakly continuous and affine and therefore define a convex compact set. Define

$$u : \mathbb{R}^{2n} \to \mathbb{R}^2 : u(x) = (\sum_{k=1}^n x_k^A, \sum_{k=1}^n x_k^B).$$

H_n is the image of $\mathcal{W}_n$ by the continuous affine map $\mu \to u\sharp\mu$, hence convex compact. Using lemma 4.1, the set $\mathcal{P}(\mu, H_n)$ is convex compact. The map $\pi \to \int\langle x, y\rangle d\pi$ is continuous on $\mathcal{P}(\mu, H_n)$ by lemma 1.4 in the appendix and the existence of a maximum follows. □

Remark 6.1: *This proof also implies that for any optimal martingale $(X, \mathcal{F})$ in the problem* (6.1)*, we can construct a multi-uniform vector U such that the pair (X_{n+1}, U) is optimal in the problem $(\mathcal{P})$. Conversely, given an optimal pair (L, U) for the problem $(\mathcal{P})$, then the associated martingale is optimal in* (6.1)*.*

Using the preceding remark and the construction described in the beginning of this section, we obtain:

Proposition 6.1: *P1 has a strategy that guarantees $\Phi_n(\mu) = \overline{\mathcal{V}_n}(\mu)$ in $\Gamma_n(\mu)$.*

Proof. Being informed of L, P1 can generate a vector $(U_k^A, U_k^B)_{k=1,\dots,n}$ such that the distribution of $(L, (U_k^A, U_k^B)_{k=1,\dots,n})$ is optimal in the maximization problem defining $\Phi_n(\mu)$. Define the associated martingale $(L_k)_{k=0,\dots,n+1}$ as in the previous lemma. Then, at round k, he plays an optimal strategy in the game $\Gamma_1([L_k \mid \mathcal{F}_{k-1}])$ (such a selection exists according to proposition 5.2). As prices posted by the two players at round k are independent of L given $\mathcal{F}_k$, the payoff for round k given $\mathcal{F}_{k-1}$ is

$$\mathbb{E}[\sum_{i=A,B} L_k^i R(p_k^i, q_k^i) + N(p_k^i, q_k^i) \mid \mathcal{F}_{k-1}]$$

P1 guarantees then $V_1([L_k \mid \mathcal{F}_{k-1}])$ at round k, and the results follows by summation. The strategy induced by this construction is admissible using remark 3.1 and proposition 5.2. Indeed, by construction, the optimal selection of P1 is such that for all k

$$\mathbb{E}[|p_k^i|] \leq 3\mathbb{E}[|L_k^i|] \leq 3\mathbb{E}[L_{n+1}^i]$$

and this last bound is independent of k. □

7. The dual game Γ_n^*

The results of this section are obtained using the same construction as for the dual game introduced in [**23**]. The main difficulty will be that we do not have anymore explicit expressions for the dual variables and for the optimal strategies of P2 in the dual game.

Duality results. At first, as a consequence of duality results from optimal transportation theory, we obtain a dual formulation of $\Phi_n(\mu)$. Recall for this the expression of Φ_n as a covariance maximization problem over probabilities with fixed marginals.

$$\Phi_n(\mu) = \max\Big\{\int \langle x,y\rangle d\pi \,\Big|\, \pi\in\mathcal{P}(\mu,H_n)\Big\}$$
$$= \underset{\nu\in H_n}{max} \quad \underset{\pi\in\mathcal{P}(\mu,\nu)}{max}\int\langle x,y\rangle d\pi$$

This problem being a special case of the Monge-Kantorovitch optimal transportation problem, we will use the following classical duality theorem (see for instance theorem 1.3 in [**57**]).

THEOREM 7.1: *(Monge-Kantorovitch) Let $(X,\mu),(Y,\nu)$ be two polish spaces equipped with some probability and $c: X\times Y\to\mathbb{R}$ a continuous function verifying:*

$$\forall (x,y)\in X\times Y,\quad c(x,y)\leqslant a(x)+b(y)$$

for some real-valued u.s.c. functions a,b such that $a\in\mathcal{L}^1(\mu)$ and $b\in\mathcal{L}^1(\nu)$.
Then the following equality holds:

$$\underset{\pi\in\mathcal{P}(\mu,\nu)}{max}\int_{X\times Y} c(x,y)d\pi(x,y) = \underset{(\phi-a,\psi-b)\in C_b(X)\times C_b(Y);\phi+\psi\geqslant c}{inf}\Big(\int_X\phi d\mu+\int_Y\psi d\nu\Big)$$

where $\phi+\psi\geqslant c$ means $\phi(x)+\psi(y)\geqslant c(x,y)$ for all $x\in X, y\in Y$ and $C_b(X)$ denotes the set of real-valued bounded continuous functions on X.

Since $(x,y)\in\mathbb{R}^2\times[-n,n]^2$, $|\langle x,y\rangle| \quad\leq\quad n(|x^A|+|x^B|)$, we can apply theorem 7.1 with $c(x,y)=\langle x,y\rangle$, $a_n(x)=n(|x^A|+|x^B|)$ and $b(x)=0$

$$\Phi_n(\mu) = \quad \max_{\nu\in H_n}\quad \inf_{(\phi-a_n,\psi)\in C_b(R^2)\times C_b([-n,n]^2);\phi+\psi\geqslant c}\Big(\int\phi d\mu+\int\psi d\nu\Big)$$

The map $((\phi,\psi),\nu)\to\int\phi d\mu+\int\psi d\nu$ is linear with respect to (ϕ,ψ), and affine with respect to ν. The set H_n is convex compact, and the set $(a_n+C_b(\mathbb{R}^2))\times C_b([-n,n]^2)$ is convex. Therefore, the minmax theorem of Sion ([**53**])

$$\Phi_n(\mu) = \inf_{(\phi-a_n,\psi)\in C_b(\mathbb{R}^2)\times C_b([-n,n]^2);\phi+\psi\geqslant c}\Big(\int\phi d\mu+\max_{\nu\in H_n}\int\psi d\nu\Big)$$

Since all the measures in H_n are concentrated $[-n,n]^2$, we will identify a function $\psi\in C_b([-n,n]^2)$ with the function defined on $\mathbb{R}^2$ which is equal to ψ on $[-n,n]^2$ and to $+\infty$ otherwise. Given a pair $(\phi-a_n,\psi)\in C_b(\mathbb{R}^2)\times C_b([-n,n]^2)$, we define as usual the convex conjugates:

$$\forall y\in\mathbb{R}^2,\ \phi^*(y) = \underset{x\in\mathbb{R}^2}{sup}\langle x,y\rangle-\phi(x)$$
$$\forall x\in\mathbb{R}^2,\ \phi^{**}(x) = \underset{y\in\mathbb{R}^2}{sup}\langle x,y\rangle-\phi^*(y)$$

The following claims are straightforward to prove using classical properties of convex conjugation when $\phi\in a_n+C_b(\mathbb{R}^2)$ (see for instance [**51**]):

i) $\phi^*\leq\psi, \phi^{**}\leq\phi$
ii) $\phi^{**}\in a_n+C_b(\mathbb{R}^2)$
iii) For all $x,y\in\mathbb{R}^2$, $\phi^{**}(x)+\phi^*(y)\geq\langle x,y\rangle$

iv) For all $\alpha \in \mathbb{R}$, $(\phi+\alpha)^* = \phi^* - \alpha$

v) ϕ^* is continuous on $[-n,n]^2$

The pair (ϕ^{**}, ϕ^*) is therefore an admissible pair in the above minimization problem. The three first properties allow us to consider only the pairs (ϕ, ϕ^*) for convex functions ϕ in $a_n + \mathcal{C}_b(\mathbb{R}^2)$. Moreover, using *iv*), we can also assume that $\phi(0) = 0$.

NOTATION 11: *D denotes the set of l.s.c. convex functions ϕ from $\mathbb{R}^2$ to $\mathbb{R}\cup\{+\infty\}$ such that $\phi(0) = 0$.*

Given $\phi \in D$, since $\phi(0) = 0$, ϕ^* is nonnegative, and $\int \phi d\mu + \sup\limits_{\nu \in H_n} \int \phi^* d\nu$ is well defined in $\mathbb{R}\cup\{+\infty\}$. We deduce from the preceding discussion:

$$
\begin{aligned}
\Phi_n(\mu) &= \inf_{\phi \in D\cap(a_n+\mathcal{C}_b(\mathbb{R}^2))} \left(\int \phi d\mu + \sup_{\nu\in H_n} \int \phi^* d\nu\right) \\
&= \inf_{\phi\in D} \left(\int \phi d\mu + \sup_{\nu\in H_n}\int \phi^* d\nu\right)
\end{aligned}
\tag{$\mathcal{D}$}
$$

DEFINITION 7.1: *For $\phi \in D$, we define:*

$$\Theta_n(\phi) = \sup_{\nu\in H_n} \int \phi^* d\nu.$$

$$\partial\Phi_n(\mu) = \{\phi \in D \,:\, \Phi_n(\mu) = \int \phi d\mu + \Theta_n(\phi)\}.$$

Note that the relation $\phi \in \partial\Phi_n(\mu)$ implies $\int \phi d\mu < +\infty$ and $\Theta_n(\phi) < +\infty$.

In order to apply this result, we have to prove the existence of minimizers for the dual problem $(\mathcal{D})$. This is the aim of the next lemma.

PROPOSITION 7.1: *For all $\mu \in \Delta^1(\mathbb{R}^2)$, there exists a solution $\phi_\mu \in D$ to the problem $(\mathcal{D})$, i.e. such that $\phi_\mu \in \partial\Phi_n(\mu)$. Moreover, if μ has compact support, the solution ϕ_μ can be chosen such that ϕ_μ^* is Lipschitz on $\mathbb{R}^2$.*

PROOF. If $\phi \in D\cap(a_n + \mathcal{C}_b(\mathbb{R}^2))$, since $\phi(0) = a_n(0) = 0$ and $\|\phi - a_n\|_\infty < \infty$, convexity of ϕ implies $|\ \phi\ | \leq a_n$. Therefore ϕ is $\sqrt{2}n$-Lipschitz. Ascoli's theorem implies then that $D\cap(a_n+\mathcal{C}_b(\mathbb{R}^2))$ is relatively compact in $C(\mathbb{R}^2)$ for the topology of uniform convergence on compact sets. Define $H(\phi) = \int \phi d\mu + \sup\limits_{\nu\in H_n} \int \phi^* d\nu$ for $\phi \in D$. Let ϕ_k be a minimizing sequence in $D\cap(a_n+\mathcal{C}_b(\mathbb{R}^2))$, we can extract a subsequence also denoted ϕ_k which converges in $\mathcal{C}(\mathbb{R}^2)$ to a limit ϕ. D being closed in $\mathcal{C}(\mathbb{R}^2)$, $\phi \in D$. Since $|\ \phi_k\ | \leq a_n$, dominated convergence implies:

$$\lim_{k\to\infty} \int \phi_k d\mu = \int \phi d\mu$$

Let K_l be a nondecreasing sequence of convex compact sets such that $\bigcup_{l\in\mathbb{N}} K_l = \mathbb{R}^2$, and ξ_{K_l} the function equal to 0 on K_l and $+\infty$ otherwise. For any function f, we define $f^{*l} = (f+\xi_{K_l})^*$, so that the sequence f^{*l} is nondecreasing and converges pointwise to f^*. Using that Fenchel transform is an isometry for the uniform norm, l being fixed, ϕ_k^{*l} converges uniformly to ϕ^{*l}

when k goes to $+\infty$. We obtain that for $\nu \in H_n$:

$$\int \phi^{*l} d\nu = \lim_{k\to\infty} \int \phi_k^{*l} d\nu \leq \liminf_{k\to\infty} \int \phi_k^* d\nu$$
$$\leq \liminf_{k\to\infty} \sup_{\nu \in H_n} \int \phi_k^* d\nu$$

Monotone convergence implies $\lim_{l\to\infty} \int \phi^{*l} d\nu = \int \phi^* d\nu$, then:

$$\sup_{\nu \in H_n} \int \phi^* d\nu \leq \liminf_{k\to\infty} \sup_{\nu \in H_n} \int \phi_k^* d\nu$$

Finally $H(\phi) \leq liminf_k H(\phi_k) = lim_k H(\phi_k)$ and $lim_k \Theta_n(\phi_k^*) = \Theta_n(\phi^*)$.
Note that if $\phi \in D$ is optimal, and μ is supported by a convex compact set K, then $\phi + \xi_K$ belongs to D and is itself optimal. Moreover $(\phi + \xi_K)^*$ is Lipschitz on $\mathbb{R}^2$. □

The dual game $\Gamma_n^*(\phi)$. This auxiliary game depends on some function $\phi \in D$. At the beginning of the game, Player 1 chooses privately the liquidation value L of the assets and then the game follows as G_n. After the n rounds of transaction, P1 has to pay a penalty of $\phi(L)$ to P2. The function ϕ is known by the players.

A behavioral strategy for player 1 is then a pair (μ, σ) with $\mu \in \Delta^1(\mathbb{R}^2)$ and $\sigma \in \Sigma^n(\mu)$. For player 2 strategies are the same as in Γ_n. The payoff function of P1 is

$$g_n^*(\phi, (\mu, \sigma), \tau) = g_n(\mu, \sigma, \tau) - \mathbb{E}_\mu[\phi(L)]$$

Since the payoff is not always well defined, we set

$$g_n^*(\phi, (\mu, \sigma), \tau) = -\infty \quad if \quad \mathbb{E}_\mu[\phi(L)] = +\infty.$$

Next proposition shows how the dual game is related to optimal behavior of P2 in the game Γ_n.

Proposition 7.2: *If $\phi \in \partial\Phi_n(\mu_0)$ and if P2 has a strategy τ^* which guarantees $\Theta_n(\phi)$ in the game $\Gamma_n^*(\phi)$, then τ^* guarantees $\Phi_n(\mu_0)$ in the game $\Gamma_n(\mu_0)$.*

Proof. τ^* guarantees $\Theta_n(\phi)$ means:

$$\forall \mu \in \Delta^1(\mathbb{R}^2), \sigma \in \Sigma_n(\mu), \quad g_n^*(\phi, (\mu, \sigma), \tau^*) \leq \Theta_n(\phi)$$

Then

$$\begin{aligned} \forall \sigma \in \Sigma_n(\mu_0), \quad g_n(\mu_0, \sigma, \tau^*) - \langle \phi, \mu_0 \rangle \leq & \quad \Theta_n(\phi) \\ \text{and finally} \quad g_n(\mu_0, \sigma, \tau^*) \leq & \quad \langle \phi, \mu_0 \rangle + \Theta_n(\phi) \\ = & \quad \Phi_n(\mu_0) \end{aligned}$$

□

Using the preceding results, in order to obtain an optimal strategy for P2 in the initial game $\Gamma_n(\mu)$, we have to show that there exists some strategy for P2 in $\Gamma_n^*(\phi)$ which guarantees $\Theta_n(\phi)$ for some $\phi \in \partial\Phi_n(\mu)$. We have the following result under an additional assumption on ϕ.

Proposition 7.3: *If $\phi \in D$ is such that ϕ^* is Lipschitz continuous on $[-n, n]^2$, then P2 has a strategy which guarantees $\Theta_n(\phi)$ in $\Gamma_n^*(\phi)$.*

We deduce then

THEOREM 7.2: *For all $\mu \in \Delta^1(\mathbb{R}^2)$, the game $\Gamma_n(\mu)$ has a value*

$$V_n(\mu) = \Phi_n(\mu).$$

and if μ has compact support, then P2 has an optimal strategy.

PROOF. Using proposition 7.1, if μ has compact support, there exists $\phi \in \partial\Phi_n(\mu)$ such that ϕ^* is Lipschitz on $\mathbb{R}^2$ and the existence of a strategy for P2 guaranteeing $\Phi_n(\mu)$ in $\Gamma_n(\mu)$ follows from propositions 7.2 and 7.3. In the general case ($\mu \in \Delta^1(\mathbb{R}^2)$), we obtain only ε-optimal strategies using the same method. Precisely, given an ε-optimal function $\phi \in a_n + \mathcal{C}_b(\mathbb{R}^2)$ in the minimization problem (D) for μ, there exists a strategy that guarantees $\Theta_n(\phi)$ in the dual game $\Gamma_n^*(\phi)$, and the same proof as in lemma 7.2 shows that this strategy guarantees $\Phi_n(\mu) + \varepsilon$ in $\Gamma_n(\mu)$. □

The proof of proposition 7.3 is based on the recursive structure of the dual game which is expressed by the following recurrence formula.

PROPOSITION 7.4: *For all $\phi \in D$*

$$\Theta_n(\phi) = \Theta_{n-1}(\Lambda(\phi^*)^*) = \Lambda^n(\phi^*)$$

where the operator Λ is defined from the set of l.s.c. convex functions from $\mathbb{R}^2$ to $\mathbb{R} \cup \{+\infty\}$ to itself by

$$\Lambda(\psi)(z) = \sup_{\nu \in H_1} \int \psi(z+w)d\nu(w).$$

PROOF. The proof follows the standard method to prove dynamic programming equations. We only prove the first equality since the second follows by induction. Note at first that for any multi-uniform vector $(U_k)_{k=1,..,n}$, the marginal law of $(U_1,..,U_{n-1})$ is in H_{n-1} and the conditional law of U_n given $(U_1,..,U_{n-1})$ is a random variable with values in H_1. Therefore

$$\mathbb{E}[\phi^*(\sum_{k=1}^n U_k)] \leq \mathbb{E}[\Lambda(\phi^*)(\sum_{k=1}^{n-1} U_k)] \leq \Theta_{n-1}(\Lambda(\phi^*)^*)$$

This implies the first inequality by taking the supremum over H_{n-1} on the left-hand side. For the reverse inequality, the mapping

$$(\nu, x) \in H_1 \times \mathbb{R}^2 \to \int \phi^*(x+w)d\nu(w) \in \mathbb{R} \cup \{+\infty\}$$

being lower semi-continuous, there exists (see theorem 2.2 in the appendix) a ε-optimal measurable selection ν_ε from $\mathbb{R}^2$ to H_1 such that

$$\int \phi^*(x+w)d\nu_\varepsilon(x)(w) \geq \begin{cases} \Lambda(\phi^*)(x) - \varepsilon & \text{if} \quad \Lambda(\phi^*)(x) < +\infty \\ \frac{1}{\varepsilon} & \text{if} \quad \Lambda(\phi^*)(x) = +\infty \end{cases}$$

Given any multi-uniform vector, $(U_k)_{k=1,..,n-1}$, one can define a random variable $U_n \in \mathbb{R}^2$ such that the conditional law of U_n given $(U_k)_{k=1,..,n-1}$ is $\nu_\varepsilon(\sum_{k=1}^{n-1} U_k)$. This construction induces

$$\Lambda(\phi^*)(\sum_{k=1}^{n-1} U_k) \leq \mathbb{E}[\phi^*(\sum_{k=1}^n U_k) \mid U_1,..,U_{n-1}] + \varepsilon$$

on the set on which $\Lambda(\phi^*)(\sum_{k=1}^{n-1} U_k) < +\infty$. The inequality is preserved by expectation if this set has always probability zero, the results follows then by sending ε to zero. If not, the left-hand side is equal to $+\infty$ and the right-hand side converges to $+\infty$ as ε goes to zero, which concludes the proof. $\square$

In order to use the strategies of P2 constructed in the one-asset model, we need the following selection lemma, which allows to reduce our problem to two separate one-dimensional problems.

LEMMA 7.1: *Let ψ be a Lipschitz convex function defined on $\mathbb{R}^2$, there exists a pair (r,t) of measurable functions from $\mathbb{R}^2 \times [-1,1]$ to $\mathbb{R}$ such that:*

– *$u \to r(x,y,u)$ and $v \to t(x,y,v)$ are convex and continuous on $[-1,1]$.*
– *$r(x,y,u) + t(x,y,v) \geq \psi(x+u, y+v) \quad \forall (x,y) \in \mathbb{R}^2,\ (u,v) \in [-1,1]^2$.*
– *$\Lambda(\psi)(x,y) = \int_{-1}^{1} r(x,y,u)\frac{du}{2} + \int_{-1}^{1} t(x,y,v)\frac{dv}{2}$.*

Moreover $\Lambda(\psi)$ is Lipschitz continuous.

PROOF. Theorem 7.1 with $c(u,v) = \psi(x+u, y+v)$ implies:

$$\Lambda(\psi)(x,y) = \underset{r,t \in C_b([-1,1]),\, r(u)+t(v) \geq \psi(x+u,y+v)}{inf} \left\{ \int_{-1}^{1} r(u)\frac{du}{2} + \int_{-1}^{1} t(v)\frac{dv}{2} \right\}$$

The pair (x,y) being fixed, define $r^{\psi^1_{x,y}}(v) = \underset{u\in[-1,1]}{sup}\ \psi(x+u, y+v) - r(u)$. Then (the proof being similar to the one for Fenchel transform)

i) $r^{\psi^1_{x,y}}$ is convex and continuous.
ii) $r^{\psi^1_{x,y}} \leq t$
iii) $\forall u, v \quad r(u) + r^{\psi^1_{x,y}}(v) \geq \psi(x+u, y+v)$
iv) For $r_1, r_2 \in C_b([-1,1])$, $\|r_1^{\psi^1_{x,y}} - r_2^{\psi^1_{x,y}}\|_\infty \leq \|r_1 - r_2\|_\infty$

Similar properties hold for $t^{\psi^2_{x,y}}(u) = \underset{v\in[-1,1]}{sup}\ \psi(x+u, y+v) - t(v)$. Let C_ψ denote the lipschit-coefficient of ψ. For $v_1, v_2 \in [-1,1]$ and $r \in C_b([-1,1])$, there exists $u_1, u_2 \in [-1,1]$ such that for $i = 1, 2$

$$r^{\psi^1_{x,y}}(v_i) = \underset{u\in[-1,1]}{sup}\ \psi(x+u, y+v_i) - r(u) = \psi(x+u_i, y+v_i) - r(u_i)$$

Therefore

$$\mid r^{\psi^1_{x,y}}(v_1) - r^{\psi^1_{x,y}}(v_2) \mid \leq \underset{i=1,2}{max} \mid \psi(x+u_i, y+v_2) - \psi(x+u_i, y+v_1) \mid \leq C_\psi \mid v_2 - v_1 \mid$$

The set $\{r^{\psi^1_{x,y}} \mid r \in C_b([-1,1])\}$ is thus uniformly equi-continuous. Due to properties $i), ii), iii)$ above, any pair (r,t) can be replaced by $((r^{\psi^1_{x,y}})^{\psi^2_{x,y}}, r^{\psi^1_{x,y}})$. Using that $(r+\alpha)^{\psi^1_{x,y}} = r^{\psi^1_{x,y}} - \alpha$, we can moreover assume that $inf_{[-1,1]}\ r = 0$. Finally

$$\Lambda(\psi)(x,y) = \underset{r\in\mathcal{B}}{inf} \left\{ \int_{-1}^{1} r(u)\frac{du}{2} + \int_{-1}^{1} r^{\psi^1_{x,y}}(v)\frac{dv}{2} \right\}$$

where $\mathcal{B}$ is the set of C_ψ-Lipschitz convex functions r defined on $[-1,1]$, such that $inf_{[-1,1]}\ r = 0$. $\mathcal{B}$ is closed in $C([-1,1],[0,2C_\psi])$ endowed with the topology of uniform convergence, and using

Ascoli's theorem, relatively compact. On the other hand, if follows from uniform continuity of ψ and property $iv)$ that the map:

$$(x,y,r) \to \int_{-1}^{1} r(u)\frac{du}{2} + \int_{-1}^{1} r^{\psi^1_{x,y}}(v)\frac{dv}{2}$$

is continuous. Therefore there exists (see theorem 2.1 in the appendix) a measurable selection f of the multivalued mapping:

$$F(x,y) = \underset{r\in\mathcal{B}}{argmin}\left\{\int_{-1}^{1} r(u)\frac{du}{2} + \int_{-1}^{1} r^{\psi^1_{x,y}}(v)\frac{dv}{2}\right\}$$

An optimal pair is then (r, r^{ψ}) with:

$$r(x,y,u) = f(x,y)(u) \qquad r^{\psi}(x,y,v) = f(x,y)^{\psi^1_{x,y}}(v)$$

The last point is obvious. □

Proof of proposition 7.3. We proceed by induction. Suppose that that for all Lipschitz convex function ϕ^*, $P2$ has a strategy $\tau^*(z)$ depending (in a measurable way) on $z \in \mathbb{R}^2$ which guarantees $\Lambda^{n-1}(\phi^*(z+.))$ in $\Gamma^*_{n-1}(\phi(.) - \langle z, .\rangle)$.

For a parameter $z_0 = (x_0, y_0) \in \mathbb{R}^2$ consider the game $\Gamma^*_n(\phi(.) - \langle z_0, .\rangle)$. Let $\tau(z_0)$ be a strategy of P2 such that $\tau_1(z_0)$ is given by $q_1(z_0) = (g^A(z_0, U), g^B(z_0, V))$ where U and V are independent uniform random variables on $[0,1]$ and for $i = A, B$, $g^i(z_0, .)$ are real-valued nondecreasing right-continuous functions defined on $(0,1)$, jointly measurable on $\mathbb{R}^2 \times (0,1)$. Let $z_1 = (x_1, y_1)$ be given by

$$x_1 = \int_0^1 (\mathbb{1}_{p_1^A > g^A(z_0,u)} - \mathbb{1}_{g^A(z_0,u) > p_1^A})du, \quad y_1 = \int_0^1 (\mathbb{1}_{p_1^B > g^B(z_0,u)} - \mathbb{1}_{g^B(z_0,u) > p_1^B})du$$

$z_1 \in [-1,1]^2$ is a measurable function of (p_1, z_0). If P2 plays after $\tau_1(z_0)$ the strategy $\tau^*(z_0 + z_1)$ for the remaining rounds we deduce that

$$\begin{aligned} g^*_n(\phi(.) - \langle z_0, .\rangle, \mu, \sigma, \tau) \leq \underset{p_1}{sup} \Bigg(& \Lambda^{n-1}(\phi^*(z_0 + z_1 + .)) \\ & + \sum_{i=A,B} \int_0^1 (\mathbb{1}_{g^i(z_0,u) > p_1^i} g^i(z_0,u) - \mathbb{1}_{p_1^i > g^i(z_0,u)} p_1^i)du \Bigg) \end{aligned}$$

To prove this inequality, just write the conditional expected payoff given p_1 and note that it is exactly the sum of a payoff in the game $\Gamma^*_{n-1}(\phi(.) - \langle z_0 + z_1, .\rangle)$ where P2 is playing $\tau^*(z_0 + z_1)$ and the second term written above. The expectation with respect to p_1 is then bounded by the supremum over p_1. Consider next the Lipschitz convex function

$$z \to \chi(z) = \Lambda^{n-1}(\phi^*(z+.))$$

Using lemma 7.1, there exists a pair of functions (r_n, t_n) such that:

$$\chi(x+u, y+v) \leq r_n(x,y,u) + t_n(x,y,v) \quad \forall (x,y) \in \mathbb{R}^2,\ (u,v) \in [-1,1]^2$$

$$\Lambda(\chi)(x,y) = \int_{-1}^{1} r_n(x,y,u)\frac{du}{2} + \int_{-1}^{1} t_n(x,y,v)\frac{dv}{2} \tag{7.1}$$

Applying this inequality with our previous result, we find

$$g_n^*(\phi(.) - \langle z_0, .\rangle, \mu, \sigma, \tau) \leq \sup_{p_1} \Bigg(r_n(x_0, y_0, x_1) + t_n(x_0, y_0, y_1) + \sum_{i=A,B} \int_0^1 (\mathbb{1}_{g^i(z_0,u)>p_1^i} g^i(z_0, u) - \mathbb{1}_{p_1^i > g^i(z_0,u)} p_1^i) du \Bigg)$$

Define :

$$g^A(x, y, u) = \frac{1}{u^2} \int_0^u 2sr_n'(x, y, 2s-1)ds \ , \quad g^B(x, y, v) = \frac{1}{v^2} \int_0^v 2st_n'(x, y, 2s-1)ds$$

Proposition 4.5 and (7.1) imply

$$g_n^*(\phi(.) - \langle z_0, .\rangle, \mu, \sigma, \tau) \leq \Lambda^n(\phi^*(z_0 + .)$$

which was exactly the desired result. □

8. From discrete to continuous time

As mentioned in section 2, the notion of price dynamics in continuous-time arising from the general class of exchange games defined in [**25**] is the result of a limit operation. To obtain such results in our context, the idea is to consider a sequence of games $\Gamma_n(\mu)$ as a discrete-time sequence approximating an heuristic continuous-time exchange game, assuming that round k occurs at time $t = \frac{k}{n}$ with $t = 0$ the date when P1 receives the message and $t = 1$ the date of public information disclosure. Since any equilibrium price martingale in $\Gamma_n(\mu)$ is a maximizer in the formulation of V_n as an optimization problem over martingales in $\mathfrak{M}_n(\mu)$, our strategy in order to obtain a limit for a sequence of equilibrium price martingales is to study first the asymptotic behavior of V_n. We present here without proofs the main ideas leading to an asymptotic expansion for V_n. Precisely, the sequence of value functions V_n divided by $\sqrt{n}$ converge to a limit V_∞ that can be expressed as the value of an optimization problem over continuous-time martingales. This result will allow us to prove some result on the convergence of sequences of maximizers for V_n (discrete-time martingales) to maximizers for V_∞.

This result being quite technical, the following presentation with our particular case of exchange mechanism is given in order to illustrate heuristically the general ideas of the general proof given in chapter 2.

Let us start with the probabilistic representation of v_n, which is the value function of the one-asset game G_n, obtained using the same proof as for proposition 5.2.

$$v_n(\mu^A) = \max_{L^A \sim \mu^A,\, (U_k^A)_{k=0,..,n} \text{ iid sequence of } \mathcal{U}_{[-1,1]}} \mathbb{E}[L^A \sum_{k=1}^n U_k^A]$$

The asymptotic behavior of v_n is not difficult to guess since it involves a sum of independent, identically distributed uniform random variables. The variance of these variables being $\rho^2 = var(\mathcal{U}_{[-1,1]}) = \frac{1}{3}$, and dividing $v_n(\mu^A)$ by $\rho\sqrt{n}$, we obtain

$$\frac{1}{\rho\sqrt{n}} v_n(\mu^A) = \max_{L^A \sim \mu^A,\, (U_k^A)_{k=0,..,n} \text{ iid sequence of } \mathcal{U}_{[-1,1]}} \mathbb{E}[L^A \frac{\sum_{k=1}^n U_k^A}{\rho\sqrt{n}}]$$

Hence, using the central limit theorem

$$\frac{1}{\rho\sqrt{n}} v_n(\mu^A) \underset{n\to\infty}{\longrightarrow} \alpha(\mu^A) = \underset{L^A\sim\mu^A, N\sim\mathcal{N}(0,1)}{max} \mathbb{E}[L^A N]$$

with $\mathcal{N}(0,1)$ the standard gaussian distribution. However, since we are interested in the asymptotic behavior of maximizers of (6.1), we need to have a more precise result. Let $(X^n, \mathcal{F}^n)$ be a sequence of maximizers for (6.1). We can assume that (see remark 6.1) these maximizers in our case are given by

- An optimal solution $(L^{A,n}, \sum_{k=1}^n U_k^{A,n})$ for the problem $v_n(\mu^A)$
- $X_k^n = \mathbb{E}[L^{A,n} \mid \mathcal{F}_k^n]$ with $\mathcal{F}_k^n = \sigma(U_1^{A,n}, .., U_k^{A,n})$ for $k = 1, .., n$ and $\mathcal{F}_{n+1}^n = \sigma(\mathcal{F}_n^n, L^A)$

The price martingale X^n is obtained by projection (conditional expectation) of $L^{A,n}$ on the filtration generated by $(U_k^{A,n})$. In order to obtain a continuous-time limit for this operation, we introduce the continuous-time processes

$$S_t^{A,n} = \frac{1}{\rho\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} U_k^{A,n} \text{ for } t \in [0,1].$$

Using a central limit theorem for martingales, the pair $(L^{A,n}, (S_t^{A,n})_{t\in[0,1]})$ converges in distribution to $(f_{\mu^A}(B_1), (B_t)_{t\in[0,1]})$ where B is a Brownian motion. However, this is not sufficient to prove that the projection property is conserved, in the sense that the continuous versions of the equilibrium price martingales X^n converge in distribution to the martingale obtained by projection in the limit

$$L_t^A = \mathbb{E}[f_{\mu^A}(B_1) \mid (B_s,\, s \le t)]$$

To prove this, the technique used in [**25**] is a functional central limit theorem obtained by embedding of the process $S_t^{A,n}$ in a Brownian filtration. The main difference there from classical limit theorems (see e.g. Kallenberg [**38**] chapter 12) is that both processes $S^{A,n}$ and X^n are embedded and this allows to deduce the asymptotic behavior of X^n from that of $S^{A,n}$.

Let us try now to adapt this procedure in the two-assets game Γ_n. Dividing the probabilistic representation of V_n given in proposition 6.1 by $\rho\sqrt{n}$, we obtain :

$$\begin{aligned}\frac{1}{\rho\sqrt{n}} V_n(\mu) &= max \mathbb{E}[\langle L, \frac{\sum_{k=1}^n U_k}{\rho\sqrt{n}}\rangle] \\ &= max \mathbb{E}[L^A \frac{\sum_{k=1}^n U_k^A}{\rho\sqrt{n}} + L^B \frac{\sum_{k=1}^n U_k^B}{\rho\sqrt{n}}]\end{aligned}$$

Since the distribution of U_k given the preceding variables is not fixed but only assumed to belong to the set H_1, a characterization of the limit distribution of the sequence

$$\frac{1}{\rho\sqrt{n}} (\sum_{k=1}^n U_k^{A,n}, \sum_{k=1}^n U_k^{B,n})$$

using some central limit result will depend on the (random) sequence of covariance matrices :

$$\begin{pmatrix} 1 & c_k^n \\ c_k^n & 1 \end{pmatrix} \text{ with } c_k^n = \frac{1}{\rho} \mathbb{E}[U_k^{A,n} U_k^{B,n} \mid U_1^n, .., U_{k-1}^n] \in [-1,1]$$

Without any additional properties of the sequence c_k^n in this formulation, we have to consider a family of possible limit distributions rather than a precise one, allowing for any continuous-time instantaneous covariance process c_t in the limit. Precisely, define the bivariate continuous-time process as above :

$$S_t^n = \frac{1}{\rho\sqrt{n}}(\sum_{k=1}^{\lfloor nt\rfloor} U_k^A, \sum_{k=1}^{\lfloor nt\rfloor} U_k^B)$$

Each sequence of coordinate processes converges in distribution to a Brownian motion. Any limiting distribution for the sequence of bivariate processes is a bi-Brownian process in the following sense: a bivariate continuous-time martingale with natural filtration $\mathcal{F}$ such that both univariate coordinate processes are Brownian motions relative to $\mathcal{F}$. Precisely

DEFINITION 8.1: *For $\omega = (\omega_1, \omega_2) \in \mathcal{C}([0,1], \mathbb{R}^2)$, let $(X_t(\omega), Y_t(\omega)) = (\omega_1(t), \omega_2(t))$ and $\mathcal{F}_t = \sigma(X_s, Y_s, s \leq t)$.*
$\mathcal{Q}$ is the set of probabilities ν on $\mathcal{C}([0,1], \mathbb{R}^2)$ such that:
1) X_t is an $(\nu, \mathcal{F}_t)$ Brownian motion.
2) Y_t is an $(\nu, \mathcal{F}_t)$ Brownian motion.
$\mathcal{Q}$ is the set of processes' distributions such that the two coordinate processes are Brownian motions with respect to the filtration $\mathcal{F}_t$.

The preceding discussion is made rigorous in chapter 2 and we deduce

THEOREM 8.1:
$$\forall \mu \in \Delta^2(\mathbb{R}^2),\ \frac{1}{\rho\sqrt{n}} V_n(\mu) \underset{n\to\infty}{\longrightarrow} V_\infty(\mu)$$
with
$$V_\infty(\mu) = sup\left\{\mathbb{E}[L^A X_1 + L^B Y_1] \,\middle|\, L \sim \mu,\ [X,Y] \in \mathcal{Q}\right\}.$$

PROOF. See theorem 11.2 and corollary 3 and remark 11.2 in chapter 2. □

The second part of the results concerning the asymptotic behavior of the price processes will be derived in the monotonic case in the next section from this probabilistic representation and in some non-monotonic cases from the dual representation of the above problem exposed in the next sections.

Before proving these convergence results, let us prove that if we define the price processes as the sequences of prices posted by P1, then the asymptotic behavior of these processes when n goes to infinity will be the same as for the abstract notion of price we have defined.

PROPOSITION 8.1: *Suppose that both players play an optimal strategy in $\Gamma_n(\mu)$, that P1's strategy is constructed using proposition 5.2 and that the sequence of continuous versions of the a posteriori martingales $(L_k^{(n)}, \mathcal{F}_k^{(n)})_{k=0,..n+1}$ converge in distribution to some process with continuous trajectories Π. Then the sequence continuous versions of posted price process defined by $\tilde{p}_0^n = \mathbb{E}(L)$ and for $t \in [0,1]$*
$$\tilde{p}_t^n = (p^{A,n}_{\lfloor nt\rfloor}, p^{B,n}_{\lfloor nt\rfloor})$$
converge in finite-dimensional distributions to the same limit Π.

PROOF. Convergence in distribution has to be understood as weak convergence in the space of probability on $\mathbb{D}([0,1], \mathbb{R}^2)$, the space of rcll (right-continuous with left-hand limits) functions endowed with the Skorokhod topology. Since the sequence Π^n converge in distribution to a limit Π having continuous trajectory, we can assume using Skorokhod's representation theorem that the processes Π^n and Π are defined on the same probability space and that the trajectories converge almost surely in uniform norm:

$$\sup_{t\in[0,1]} \|\Pi^n_t - \Pi_t\| \underset{n\to\infty}{\longrightarrow} 0 \ a.s.$$

The variables $(\Pi^n_t, \Pi_t, n \in \mathbb{N}, t \in [0,1])$ being uniformly integrable, we deduce for all $t > 0$ the following L^1 convergences :

$$\mathbb{E}[\|\Pi^n_t - \Pi_t\|] \underset{n\to\infty}{\longrightarrow} 0 \quad , \quad \mathbb{E}[\left\|\Pi^n_t - \Pi^n_{t-\frac{1}{n}}\right\|] \underset{n\to\infty}{\longrightarrow} 0$$

From the construction of P1 optimal strategies and the last assertion in lemma 4.4, we have for all n,$k = 1, .., n$ and $i = A, B$ [3]

$$\mathbb{E}[|L^{i,n}_k - p^{i,n}_k|] \leq \mathbb{E}[|L^{i,n}_k - L^{i,n}_{k-1}|]$$

Therefore, for all t

$$\begin{aligned}\mathbb{E}[\|\tilde{p}^n_t - \Pi_t\|] &\leq \mathbb{E}[\|\Pi^n_t - \Pi_t\|] + \mathbb{E}[\|\Pi^n_t - \tilde{p}^n_t\|] \\ &\leq \mathbb{E}[\|\Pi^n_t - \Pi_t\|] + \mathbb{E}[\left\|\Pi^n_t - \Pi^n_{t-\frac{1}{n}}\right\|] \underset{n\to\infty}{\longrightarrow} 0\end{aligned}$$

and this implies the convergence in finite-dimensional distributions of $\tilde{p}^n$ to Π. □

9. A Monotonic derivative

We focus in this section on the first question addressed in the introduction concerning the robustness of the one asset model against the introduction of a derivative, namely how the introduction of the asset B affects the price dynamics of A. To answer, we compare the dynamics obtained in the one-asset model for A to the dynamics obtained for A, B in our model, when in the two models, A has the same marginal distribution. As mentioned in the introduction, if the two assets we consider are linked, actions taken by the informed agent Player 1 on market B are relevant for the uninformed agent to trade on market A. Therefore, in general, the value of the two-assets game is less than the sum of the two corresponding one-asset games, Player 2 having more information in the two-assets game. The price dynamic will be said to be robust against the introduction of a derivative B if the asymptotic behavior of the price processes for the asset A is the same in the two models. If we assume that this derivative is monotonic, which in our context means that μ is such that L^B is a monotonic function of L^A with probability 1, then we will prove that the price dynamic is robust. A typical example is when B is an european put or call option on A.

Actually, we can solve explicitly the optimization problems V_n and V_∞ in this case. The proof relies on the notion of comonotonicity and its relation with the one-dimensional Monge-Kantorovitch problem already discussed in lemma 4.2.

3. Using the construction of the optimal selection in proposition 5.2, we can assume that the variable $p^{i,n}_k$ are defined on the same probability space up to an enlargement.

At first, note that for any probability μ, the following inequality holds

$$V_n(\mu) \leq v_n(\mu^A) + v_n(\mu^B)$$

since relaxing the constraint $L \sim \mu$ (in the formulation defining Φ_n) into the pair of constraints $L^A \sim \mu^A$ and $L^B \sim \mu^B$ transforms the left-hand side into the right-hand side.

In order to obtain an equality one must construct a vector $(L, S) \in \mathbb{R}^2 \times \mathbb{R}^2$ such that each pair (L^i, S^i) is comonotonic for $i = A, B$ and fulfilling the marginal constraints $L \sim \mu$ and $[S] \in H_n$. This is in general not possible, but if μ is such that $L^B = g(L^A)$ for some nondecreasing function g, then the pair L^A, L^B is itself comonotonic. This notion is transitive, we can construct a comonotonic pair (L^A, S^A) with the prescribed distribution and define $L^B = g(L^A)$ so that $L \sim \mu$. By composition of nondecreasing functions, (L^B, S^A) is comonotonic, and $[S^A, S^A] \in H_n$. The above inequality is therefore an equality and the optimal solution depends only on one variable S^A

$$V_n(\mu) = v_n(\mu^A) + v_n(\mu^B) \tag{9.1}$$

The main result in this section is then theorem 9.2, which implies that the asymptotic distribution of the equilibrium price process of the asset A in not affected by the introduction of a monotonic derivative in the game.

We assume that $\mu \in \Delta^2(\mathbb{R}^2)$ is such that $L^B = g(L^A)$ μ-a.s. for some nondecreasing function g. First recall the results obtained in [**25**]:

THEOREM 9.1:

$$\forall \mu^A \in \Delta^2(\mathbb{R}),\ \frac{1}{\rho\sqrt{n}} v_n(\mu^A) \underset{n\to\infty}{\longrightarrow} \rho\alpha(\mu^A)$$

with $\alpha(\mu^A) = \underset{\pi\in\mathcal{P}(\mu^A,\mathcal{N}(0,1))}{sup} \int xy d\pi(x,y) = \mathbb{E}[f_{\mu^A}(N)N]$ *for some r.v.* $N \sim \mathcal{N}(0,1)$.

PROOF. See [**25**]. □

With the notations of the previous section, we have:

$$V_\infty(\mu) = sup\left\{\mathbb{E}[L^A X_1 + L^B Y_1] \;\middle|\; L \sim \mu,\ [X, Y] \in \mathcal{Q}\right\}$$

Since X_1 and Y_1 are standard gaussian r.v. and $L \sim \mu$, we deduce from the definition of α that:

$$V_\infty(\mu) \leq \alpha(\mu^A) + \alpha(\mu^B)$$

Given a Brownian motion X, we define $L^A = f_{\mu^A}(X_1)$ so that $L^B = g(f_{\mu^A}(X_1))$ is a nondecreasing function of X_1. This implies $g \circ f_{\mu^A} = f_{\mu^B}$ and the process (X, X) is an optimal bi-Brownian process since

$$\mathbb{E}[L^A X_1 + L^B X_1] = \mathbb{E}[f_{\mu^A}(X_1)X_1 + f_{\mu^B}(X_1)X_1] = \alpha(\mu^A) + \alpha(\mu^B)$$

and we recover finally the following inequality, which could be deduced from (9.1), and theorems 9.1 and 8.1 by sending n to infinity.

$$V_\infty(\mu) = \alpha(\mu^A) + \alpha(\mu^B). \tag{9.2}$$

Suppose that both players play an optimal strategy in $\Gamma^n(\mu)$. Then the price martingale $(L^n_k, \mathcal{F}^n_k)_{k=0,..n+1}$ must solve the problem (6.1):

$$\mathbb{E}[\sum_{k=1}^{n} V_1([L^n_k \mid \mathcal{F}^n_{k-1}])] = V_n(\mu)$$

Let B be a standard Brownian motion defined on $(\Omega, \mathcal{A}, \mathbb{P})$ and let $(\mathcal{G}_t)_{t\geq 0}$ denotes its natural filtration. Consider next the bivariate process $\Pi^\mu = (\Pi^{\mu^A}, \Pi^{\mu^B})$ defined as in the introduction by:

$$\Pi^{\mu^A}_t = \mathbb{E}[f_{\mu^A}(B_1) \mid \mathcal{G}_t] \ , \ \Pi^{\mu^B}_t = \mathbb{E}[f_{\mu^B}(B_1) \mid \mathcal{G}_t]$$

This process is actually the unique comonotonic martingale coupling of the CMMV martingales of final distribution μ^A and μ^B.

Theorem 9.2: *If both players play an optimal strategy in $\Gamma^n(\mu)$, then the continuous versions of the price martingales $(L^{(n)}_k, \mathcal{F}^{(n)}_k)_{k=0,..n+1}$ converge in distribution to the process Π^μ.*

Proof. Using theorem 1.2 in chapter 2, we know that the sequence of laws of the continuous-time versions of the martingales $(L^{(n)}_k, \mathcal{F}^{(n)}_k)_{k=0,..n+1}$ is relatively compact for the Meyer-Zheng topology. Moreover, any limit belongs to the set $\mathcal{P}_\infty(\mu)$ that we define now. At first, we have

$$V_\infty(\mu) = \underset{[(L_t,Z_t)_{t\in[0,1]}]\in\mathcal{M}(\preceq_\mu,\mathcal{Q})}{max} \mathbb{E}[\langle L_1, Z_1\rangle]$$

where $\mathcal{M}(\preceq \ \mu, \mathcal{Q})$ is the set of martingales distributions of processes $(L_t, Z_t)_{t\in[0,1]}$ in $\Delta(\mathbb{D}([0,1], \mathbb{R}^4))$ (with $\mathbb{D}$ the set of cádlág functions) such that $[L_1] \preceq \mu$ and $[(Z_t)_{t\in[0,1]}] \in \mathcal{Q}$ (using the identification of the continuous functions as a subset of $\mathbb{D}$). The set of image probabilities by the projection on the first coordinate $(L_t)_{t\in[0,1]}$ of the set of maximizers will be denoted $\mathcal{P}_\infty(\mu)$. This set is simply the generalization of the law of the martingale L^A_t obtained by projection on the Brownian filtration in the unidimensional case. Using the previous discussion, if a martingale $(L_t, Z_t)_{t\in[0,1]}$ is optimal in the previous problem, then necessarily, we have $L^A_1 = f_{\mu^A}(Z^A_1)$ and $L^B_1 = f_{\mu^B}(Z^A_1)$. By definition, the process $(L_t, Z_t)_{t\in[0,1]}$ is a martingale with respect to the filtration it generates $\mathcal{F}_t = \sigma(L_s, Z_s, s \leq t)$. But Z^A is a Brownian motion and a $\mathcal{F}_t$-martingale and therefore an $\mathcal{F}_t$-Brownian motion (see 3.3.16 in [**39**]). It follows that

$$L^A_t = \mathbb{E}[L^A_1 \mid \mathcal{F}_t] = \mathbb{E}[f_{\mu^A}(Z^A_1) \mid \mathcal{F}_t] = \mathbb{E}[f_{\mu^A}(Z^A_1) \mid (Z_s, s \leq t)]$$

As the same computation holds for L^B, this shows that $(L_t)_{t\in[0,1]}$ has a continuous version which follows the law of the martingale Π^μ. Since $(L_t)_{t\in[0,1]}$ is a cádlág martingale, its law is therefore equal to the law of Π^μ. In order to conclude the proof, the convergence in law of a sequence of uniformly integrable martingales to a martingale with continuous trajectories for the Meyer-Zheng topology implies the convergence for the Skorokhod topology as shown in [**48**] (the exact uniform integrability required to apply this result is a direct consequence of Doob's inequality and the fact that our martingales have uniformly bounded second order moments). □

Remark 9.1: *All this discussion can be easily adapted to the anti-monotonic case by replacing (X, X) by $(X, -X)$ in the proof of* (9.2). *Theorem 9.2 holds with $(f_{\mu^A}(B_1), f_{\mu^B}(-B_1))$ instead of $(f_{\mu^A}(B_1), f_{\mu^B}(B_1))$.*

10. The dual problem and a non-monotonic example.

A general dual formulation of V_∞ as a stochastic control problem is given in chapter 2. We will solve this problem in a very particular case, corresponding the case of non-monotonic derivative, providing an explicit example of price dynamics which differs from the class CMMV.

The dual Problem. Recall the asymptotic problem obtained in section 1.2 written in integral form

$$V_\infty(\mu) = \underset{\nu\in\mathcal{Q}_1}{max}\ \underset{\pi\in\mathcal{P}(\mu,\nu)}{max} \int \langle u,v\rangle d\pi(u,v)$$

Applying theorem 7.1 $c(x,y) = \langle x,y\rangle$ and the minmax theorem, the following result is proposition 7.1 in chapter 2

PROPOSITION 10.1:

$$V_\infty(\mu) = \underset{\phi}{min}(\mathbb{E}_\mu[\phi(L)] + \underset{\nu\in\mathcal{Q}(1)}{max}\mathbb{E}_\nu[\phi^*(X_1,Y_1)]) \tag{10.1}$$

where the minimum is taken and exists in the set of proper l.s.c. convex functions from $\mathbb{R}^2$ to $\mathbb{R}\cup\{+\infty\}$.

The dual problem is then defined by

$$V^*_\infty(\phi) = \max_{\nu\in\mathcal{Q}(1)} \mathbb{E}_\nu[\phi^*(X_1,Y_1)]$$

$V^*_\infty(\phi)$ is associated to a stochastic control problem and the value function of this problem starting from a point (x,y) at time t is by definition:

$$u(x,y,t) = \sup_{\mathbb{P}\in\mathcal{Q}} \mathbb{E}_{\mathbb{P}}[\phi^*(x+X_{1-t}, y+Y_{1-t})]$$

This function is the unique (continuous) viscosity solution of the HJB equation (proposition 8.1 in chapter 2)

$$\begin{cases} -u_t = \frac{1}{2}\Delta u + \mid u_{xy}\mid & \text{for } (x,y,t)\in\mathbb{R}^2\times[0,1) \\ u(x,y,1) = \psi^*(x,y) & \text{for } (x,y)\in\mathbb{R}^2 \end{cases} \tag{10.2}$$

for all function ϕ^* in the class of functions $\mathfrak{C}$ with the following restriction on growth:

$$u\in\mathfrak{C} \Longleftrightarrow \forall t\in(0,1),\ \exists M,\rho>0,\ \forall(s,x)\in[t,1]\times\mathbb{R}^2,\ |u(s,x)|\le Me^{\rho|x|^2}$$

The next result summarizes the relation between the solutions of the primal and dual problems.

LEMMA 10.1: *In the following, μ denotes the law of the variable L in $\Delta^2(\mathbb{R}^2)$ and $Z=(X,Y)$ is a process whose law is in $\mathcal{Q}$, both defined on the same probability space. The two following assertions are equivalent*

$$i) L\in\partial\phi^*(Z_1) \textit{ almost surely, and } \mathbb{E}[\phi^*(Z_1)] = \underset{\nu\in\mathcal{Q}(1)}{sup}\langle\nu,\phi^*\rangle$$

ii) The joint distribution of (L,Z) is optimal for $V_\infty(\mu)$

and ϕ is optimal in the dual formulation (10.1).

The HJB equation associated to this problem is degenerate, and general existence theorems for a classic $C^{2,1}$-solution to HJB equations do not apply since they rely on a strict ellipticity condition. However, in the following example we can provide a $C^{2,1}$ solution to the equation (10.2).

A basic non-monotonic derivative : The three points case. We apply now the preceding results to the family of distributions C concentrated on the three points

$$I = (-1, 1); J = (0, 0); K = (1, 1)$$

Note that if $L \sim \mu \in C$, we have $L^B = \mid L^A \mid$, and B is then a non monotonic derivative on A.

According to lemma 10.1, in order to obtain a solution to $V(\mu)$ for $\mu \in C$, we focus on the family of convex function ϕ such that $\nabla\phi^*$ is valued in $\{I, J, K\}$ (at least almost surely for the optimal distribution of the problem $V^*(\phi)$). We consider the family $\gamma^*(x + ., y + .)$, for $(x, y) \in \mathbb{R}^2$ with $\gamma^*(u, v) = (v + |u|)^+$. Up to a translation, the value function associated to these functions is always :

$$u(x, y, t) = \max_{\mathbb{P}\in\mathcal{Q}} \mathbb{E}[\gamma^*(x + X_{1-t}, y + Y_{1-t})]$$

Using the scaling property of Brownian motion (processes (X_t) and $(\frac{1}{c}X_{c^2t})$ have same law) and the positive homogeneity of γ^*, we deduce

$$u(x, y, t) = \sqrt{1-t}\; u(\tfrac{x}{\sqrt{1-t}}, \tfrac{y}{\sqrt{1-t}}, 0)$$

With $z = (x, y)$ and $G(z) = u(z, 0)$, this leads to the equation:

$$G(z) = \langle\nabla G(z), z\rangle + \Delta G(z) + 2 \mid G_{xy}(z) \mid \tag{10.3}$$

An explicit solution. Using an heuristic argument, we construct now an explicit solution to the above problem.

The function γ^* is convex and linear outside the set

$$Z = \{y = - \mid x \mid\} \cup \{y \geq 0, x = 0\}.$$

Intuitively, in order to maximize the above expectation, the process we control has to pass through Z a "maximal number of times". Using the heuristic approach explained in section 8, at time t, we have to choose c_t, that is to choose on which of the two diagonals in the plan we want to direct (X, Y) starting from (X_t, Y_t). A reasonable choice is the diagonal that minimizes the distance to Z, and this leads to $c_t = sgn(X_t)$.

The process defined by:

$$\begin{cases} dX_t = dB_t, & X_0 = x \\ dY_t = sgn(X_t)dB_t, & Y_0 = y \end{cases}$$

where B is a standard Brownian motion seems to be a good candidate.

The law of the process we just defined is well-known and related to the local time of Brownian motion. We can therefore check by a direct computation (reproduced in section 12) that the function

$$G(x, y) = \mathbb{E}[\gamma^*(X_1, Y_1)]$$

is a C^2 solution of the equation (10.3). If we define

$$\overline{u}(x,y,t) = \sqrt{1-t}\; G(\tfrac{x}{\sqrt{1-t}}, \tfrac{y}{\sqrt{1-t}})$$

then $\overline{u}$ is the unique solution of (10.2) and $c(x,y) = sgn(x)$ is an optimal markovian control in the sense that

$$\overline{u}_t(x,y,t) = \tfrac{1}{2}\Delta\overline{u}(x,y,t) - c(x,y)\overline{u}_{xy}(x,y,t) = \tfrac{1}{2}\Delta\overline{u} - \mid \overline{u}_{xy} \mid$$

We deduce from theorem 1.3 that (X,Y) is the unique maximizer (in law) of the problem

$$\max_{\mathbb{P}\in\mathcal{Q}} \mathbb{E}[\gamma^*(x+X_1, y+Y_1)]$$

Indeed, any maximizer $(\widetilde{X},\widetilde{Y})$ should be such that

$$\frac{d}{dt}(\langle\widetilde{X},\widetilde{Y}\rangle_t) = sgn(\widetilde{X}_t) \quad dt\otimes d\mathbb{P} \text{ a.s.}$$

since $\mathbb{E}[\int_0^1 \mathbb{I}_{\widetilde{X}_s=0} ds] = 0$. Since

$$\langle\int_0^\cdot sgn(\widetilde{X}_s)d\widetilde{X}_s, \widetilde{Y}\rangle_t = t$$

we deduce that

$$\widetilde{Y}_t = \int_0^t sgn(\widetilde{X}_s)d\widetilde{X}_s$$

which determines the law of law $(\widetilde{X},\widetilde{Y})$.

For (x,y), the distribution of $(X_1,Y_1) = (x+B_1, y+\int_0^1 sgn(x+B_s)dB_s)$ gives probability zero to the set Z where γ^* is not linear. Then the random variable $L = \nabla\gamma^*(X_1,Y_1)$ is well defined and its distribution μ belongs to C. Moreover, we have by a direct computation

$$\mathbb{E}_\mu[L] = \mathbb{E}[\nabla\gamma^*(X_1,Y_1)] = (\frac{\partial\overline{u}}{\partial x}(x,y,0), \frac{\partial\overline{u}}{\partial y}(x,y,0))$$

We can check that $(\frac{\partial\overline{u}}{\partial x}(x,y,0), \frac{\partial\overline{u}}{\partial y}(x,y,0))$ is bijective from $\mathbb{R}^2$ to the interior of the triangle IJK. Denoting $int(C)$ the relative interior of C, we conclude

PROPOSITION 10.2: *$\forall\mu\in int(C)$, $\exists(x,y)\in\mathbb{R}^2$ such that :*

$$L := \nabla\gamma^*(x+B_1, y+\int_0^1 sgn(x+B_s)dB_s) \sim \mu$$

$$V(\mu) = \mathbb{E}[L^A B_1 + L^B\int_0^1 sgn(x+B_s)dB_s]$$

where (x,y) is the unique solution of

$$(\frac{\partial\overline{u}}{\partial x}(x,y,0), \frac{\partial\overline{u}}{\partial y}(x,y,0)) = \mathbb{E}_\mu[L]$$

Existence of a smooth solution implies actually a much stronger result.

THEOREM 10.1: *For all $\mu \in int(C)$, any sequence of continuous-time versions of martingales in $\mathfrak{M}_n(\mu)$, asymptotically optimal for the problem* (6.1) *converges in distribution to the continuous bivariate martingale*

$$L_t^A = \mathbb{E}[\nabla_x \gamma^*(X_1, Y_1) \mid \mathcal{G}_t] = \frac{\partial \overline{u}}{\partial x}(X_t, Y_t, t)$$
$$L_t^B = \mathbb{E}[\nabla_y \gamma^*(X_1, Y_1) \mid \mathcal{G}_t] = \frac{\partial \overline{u}}{\partial y}(X_t, Y_t, t)$$

This applies in particular for any sequence of optimal price processes in $\Gamma_n(\mu)$.

PROOF. It is sufficient to show that the set $\mathcal{P}_\infty(\mu)$ is reduced to a single point, which is the law of the martingale $(L_t)_{t\in[0,1]}$ defined in the theorem. This follows from the preceding discussion and theorem 10.1 in chapter 2 which implies that any optimal martingale $\widehat{L}$ must verify the following equalities for $i = A, B$ and $t \in [0, 1]$

$$L_t^i = \frac{\partial \overline{u}}{\partial x}(X_t, Y_t, t)$$

□

We deduce from this result that $V(\mu) < \alpha(\mu^A) + \alpha(\mu^B)$ for $\mu \in int(C)$, otherwise we should have the same equalities as in section between L_1^A and X_1. This means that the asymptotic behavior of the value of the game has been modified by the introduction of this particular non monotonic derivative.

Non-uniqueness. We end this section with a simple negative result. Consider the gaussian bivariate distribution $\mu = \mathcal{N}(0, Id)$. If $L \sim \mu$, then the two assets are independent and our game is separable, in the sense that it can be reduced to a pair of one-asset games played in parallel. However, the problem $V(\mu)$ has an infinity of maximizers. The most intuitive is to consider a bi-dimensional Brownian process (X, Y) and $L = (X_1, Y_1)$. But, we can construct a bi-Brownian process (X, Y) having the same terminal distribution. Consider a Brownian motion X and define $Y_t = \int_0^t c_s dB_s$ with c_s the deterministic function equal to 1 on $[0, 1/2]$ and to -1 on $(1/2, 1]$. We have $Y_1 = X_{\frac{1}{2}} - (X_1 - X_{\frac{1}{2}})$ and this implies clearly $(X_1, Y_1) \sim \mu$ using the characterization of X as gaussian process. Using the stability of gaussian random variables, any deterministic measurable function c_s with values in $\{-1, 1\}$ and such that $\int_0^1 c_s ds = 0$ leads to the same result. We have therefore an infinity of possible price dynamics. The same construction holds for any bivariate gaussian distribution with some correlation coefficient $r \in (-1, 1)$ replacing the condition on c_s by $\int_0^1 c_s ds = r$. This example shows that the uniqueness of maximizers for the problem V fails in general.

11. Some Extensions

All the asymptotic results can be generalized in a large class of games as in [**25**] using the results obtained in chapter 1.

Games in this class will still be denoted by $\Gamma_n(\mu)$ since they differ only from the game we have studied in the earlier sections by the exchange mechanism T. It is no more possible to prove existence of the value or of optimal strategies without specifying T. We will therefore make similar hypotheses on the trading mechanism as in [**25**]. In order to avoid technical difficulties, let us assume that T is bounded. (see [**25**] for possible extensions). We assume H1,H4 and H5 on $\Delta^2(\mathbb{R})$ as in section 2, and the following two hypotheses

H1') *Existence of a value*: For all n, for all $\mu \in \Delta^2(\mathbb{R}^2)$, $\Gamma_n(\mu)$ has a value $V_n(\mu)$ and both players have optimal strategies.

H3') *Symmetry and Invariance with respect to the numéraire scale*

$$\forall \alpha \in \mathbb{R},\ \forall X \in L^2, \quad v_1([\alpha X]) = \mid \alpha \mid v_1([X])$$

Proposition 1.2 in chapter 1 and theorem 11.2 in chapter 2 show that the asymptotic analysis of any game in this class will be equivalent to the analysis made for our particular game. Since the asymptotic behavior of V_n and of the optimal price processes is the same within all this class, this results strengthens the result obtained in [**25**] in the sense that the class CMMV is robust against the introduction of monotonic derivatives in a large class of models.

REMARK 11.1: *We did not assume $H2$ since it always hold for bounded mappings T. In case T is unbounded, one has to define a notion of admissible strategies as we did in section 3. $H2$ allows in this case to weaken the Lipschitz assumption on the value.*

12. Complements.

Computation of $\mathbb{E}[\gamma^*(X_1, Y_1)]$. The law of the process X, Y is well-known and related to the local time of Brownian motion. Define $\beta_t = \int_0^t sgn(X_s)dB_s$ and $S_t = \underset{s\leq t}{inf}\beta_s$, and let L_t denote the local time of X in 0. Tanaka's formula (see [**50**] chapter VI) gives

$$\mid X_t \mid = \mid x \mid + \beta_t + L_t$$

This implies

$$S_t = \underset{s\leq t}{inf}\beta_s = \underset{s\leq t}{inf} \mid X_s \mid - \mid x \mid - L_s$$

Using properties classical of the local time of X, we deduce that

$$\begin{cases} \mid X_t \mid = \mid x \mid + \beta_t & \text{if} \quad S_t > - \mid x \mid \\ \mid X_t \mid = \beta_t - S_t & \text{if} \quad S_t \leq - \mid x \mid \end{cases}$$

and finally:

$$\gamma^*(X_1, Y_1) = (Y_1 + \mid X_1 \mid)^+ = \begin{cases} (y + 2\beta_1 + \mid x \mid)^+ & if \quad S_1 > - \mid x \mid \\ (y + 2\beta_1 - S_1)^+ & if \quad S_1 \leq - \mid x \mid \end{cases}$$

The distribution of (β_1, S_1) is known (see proposition 2.8.1 in [**39**]) and given by the following density on $\mathbb{R}^2$:

$$f(s,b) = \tfrac{2(b-2s)}{\sqrt{2\pi}} exp(\tfrac{-(b-2s)^2}{2}) \mathbb{1}_{(s\leq 0, b\geq s)}$$

We compute:

$$\begin{aligned}
G(x,y) &= \mathbb{E}[\gamma^*(X_1,Y_1)] \\
&= \int_{-|x|}^{0} ds \int_{\frac{-y-|x|}{2}}^{+\infty} (y+2b+\mid x\mid) f(s,b)db \\
&\quad + \int_{-\infty}^{-|x|} ds \int_{\frac{s-y}{2}}^{+\infty} (y+2b-s) f(s,b)db
\end{aligned}$$

The function G is $\mathcal{C}^2$ an its derivatives are given by:

$$G_x(x,y) = +sgn(x)\int_{-|x|}^{0} ds \int_{\frac{-y-|x|}{2}}^{+\infty} f(s,b)db$$

$$G_y(x,y) = \int_{-|x|}^{0} ds \int_{\frac{-y-|x|}{2}}^{+\infty} f(s,b)db + \int_{-\infty}^{-|x|} ds \int_{\frac{s-y}{2}}^{+\infty} f(s,b)db$$

$$G_{xx} = F_1 + F_2, \quad G_{yy} = F_2 + F_3, \quad G_{xy} = +sgn(x)F_2$$

with:

$$\begin{aligned}
F_1(x,y) &= \tfrac{2}{\sqrt{2\pi}}\Big[exp(-\tfrac{1}{2}x^2)\mathbb{1}_{(y\geq|x|)} + exp(-\tfrac{1}{2}(\tfrac{+3|x|-y}{2})^2)\mathbb{1}_{(y<|x|)}\Big] \\
F_2(x,y) &= \tfrac{1}{2\sqrt{2\pi}}\Big[exp(-\tfrac{1}{2}(\tfrac{y+|x|}{2})^2) - exp(-\tfrac{1}{2}(\tfrac{3|x|-y}{2})^2)\Big]\mathbb{1}_{(y\leq|x|)} \\
F_3(x,y) &= \tfrac{2}{3\sqrt{2\pi}}\Big[exp(-\tfrac{1}{2}y^2)\mathbb{1}_{(y\geq|x|)} + exp(-\tfrac{1}{2}(\tfrac{3|x|-y}{2})^2)\mathbb{1}_{(y<|x|)}\Big]
\end{aligned}$$

and we check that G is solution of the equation (10.3).

CHAPITRE 4

Etude asymptotique d'un jeu d'échange à somme non-nulle.

On analyse dans ce chapitre un jeu d'échange à somme non-nulle avec information incomplète d'un côté. Dans les sections 2 à 5, on définit un jeu dual et on prouve l'existence d'une classe particulière d'équilibres dans le jeu dual appelés équilibres réduits et auto-égalisateurs.

Dans la section 6, on étudie le comportement asymptotique de ces équilibres, on fait apparaître en particulier un processus de prix limite en temps continu.

On prouve enfin dans la section 7 un résultat d'existence d'équilibre dans le jeu initial pour une classe de lois régulières.

1. Introduction

Brownian motion and related continuous martingales are used in many financial models to describe the stock prices. The appearance of these processes is often explained exogenously: prices depends on a long list of external random parameters whose effects aggregate in a Brownian motion due to an implicit central limit theorem. In "On the strategic origin of Brownian Motion in finance" [**31**], an endogenous explanation is provided : the Brownian Motion is partially introduced by the agents to maximize their profit. The main idea is to consider informational asymmetries. Suppose that an informed agent is trading with an uninformed one, who just knows that his opponent is informed. Each move of the insider on the market is analyzed by the other to infer its informational content. A naive use of information would disclose it and induce a loss of this strategic advantage for the next periods. At each transaction the informed agent has thus to care about how much information his action will reveal. In order to take benefit of his information, he has to find how to control the rate of revelation. As proved in [**31**], his optimal behavior is to introduce some random noises in his actions, and these noises in the day after day transactions aggregate in a Brownian motion. To illustrate this idea, the authors consider a n times repeated exchange game between asymmetrically informed risk-neutral agents. They show that any sequence of equilibrium price processes converges in distribution, as n goes to infinity, to a particular Brownian martingale. This model as well as the following generalizations ([**25**]) were dealing with zero-sum games. In order to extend these results to a risk-aversion setting or to m-player games, we have to work with non-zero-sum games. We aim to prove in this work that Brownian dynamics also appear in a particular non-zero-sum market game.

The organization of the paper is as follows: At first, we describe the model and state the main result in section 1. In section 3, we introduce an auxiliary game, which is a generalization of the dual game introduced in [**23**] for repeated zero-sum games with incomplete information. In particular, we show how the equilibria in this dual game are related to the equilibria in our initial game, extending this way in the nonzero-sum case some results of [**23**]. Sections 4 and 5 are devoted to prove the existence of an equilibrium in the dual game, which is the unique one in a particular class of equilibria defined in section 5, called reduced and self-equalizing equilibria. In section 6, we prove that the sequence of price processes associated to these equilibria converges to some Brownian martingale as n goes to infinity, and we deduce from this result an asymptotic result on the price processes in the initial game.

2. The Model

The game $G_n(\mu)$. In this game $G_n(\mu)$, two market makers (Player 1 and 2) are trading a risky asset R against a numéraire N. At the beginning of the game, Player 1 receives some private message concerning asset R. At a future date ($t = 1$), say at the next shareholder meeting, P1's message will be publicly disclosed. At that date, the price L of asset R is called the liquidation value of R. It will depend on P1's message. Since L is the only useful content of P1's information, we may assume that nature initially chooses L with a lottery $\mu \in \Delta(\mathbb{R})$

(where $\Delta(\mathbb{R})$ denotes the set of Borelian probabilities on $\mathbb{R}$), informs P1 but not P2 who only knows μ.

Before the disclosure date $t = 1$, there are n consecutive trading rounds. As market makers, players 1 and 2 are committed to post, at round k, prices p_k, q_k for the risky asset R. A price p posted by player 1 is a commitment to buy or sell one share of the risky asset R against p units of N. Prices are posted independently and simultaneously, and publicly observed after each round. Clearly, if $p_k \neq q_k$, a trader will see an arbitrage possibility, and he will buy at the lowest price one share of R to sell it immediately at the highest price. The external trader we introduce in this model is not a strategic player. At round k ($k = 1, ..., n$), the players have to select a pair of actions (p_k, q_k) independently of each others, based on their past observations and private information. Next, the trader realizes the arbitrage if there is any and the actions are announced publicly.

If $y_k^n = (y_k^R, y_k^N)$ and $z_k = (z_k^R, z_k^N)$ denote P1's and P2's portfolio after round k, then:

$$\begin{aligned} y_{k+1} &= y_k + sg(p_k - q_k)(1, -p_k) \\ z_{k+1} &= z_k + sg(p_k - q_k)(-1, q_k) \end{aligned}$$

$$\text{where} \quad sg(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$$

Both players are assumed to have sufficiently large initial endowments, and therefore the constraints $y_k \geq 0$ and $z_k \geq 0$ can be ignored in this model. The players are supposed to be risk neutral so that the utility they aim to maximize is the expected value of their final portfolio. P1'utility is: $\mathbb{E}[y_n^R L + y_n^N]$. Since y_0 is initially fixed, its liquidation value is independent of players' moves. It can thus be subtracted from P1's utility without affecting his behavior in the game. The same argument can be applied to P2 and this amounts to assume $y_0 = z_0 = (0, 0)$.

Players are assumed to use behavioral strategies: at each round, they can use a lottery to select their actions (see next section for a precise description). Therefore, at equilibrium in $G^n(\mu)$, the sequence of prices posted by P1, $(p_k^n)_{k=1,..,n}$ is a stochastic process. The n transaction rounds occur at time $t = \frac{k}{n}$, and we can then extend this price process in a continuous time process Π_t^n, piecewise constant on the intervals $[\frac{k}{n}, \frac{k+1}{n})$. Precisely, with $\lfloor a \rfloor$ the greater integer less or equal to a, and $p_0^n = \mathbb{E}[L]$, we have :

$$\text{For } t \in [0, 1], \quad \Pi_t^n = p_{\lfloor nt \rfloor}^n$$

When n becomes large, then the time between two transactions tends to zero, and the price process appears naturally as an approximation of a continuous-time price process. The limit, if it exists, of the processes Π_t^n, can be interpreted as a continuous-time "equilibrium" price process.

Previous results in [**31**] and [**25**] show that in the zero-sum case, for any sequence of equilibria, this limit exists and is independent of the chosen sequence and of the exchange mechanism of the game. It is a Brownian martingale called continuous martingale of maximal variation that depends only of the distribution μ of the liquidation value of the risky asset.

Main result. The aim of this work is to obtain a asymptotic result in the same spirit as in the zero-sum case. However, since many equilibria with different payoffs could exist in the nonzero-sum setting, we will focus on a particular class of equilibria called reduced and self-equalizing. For this class, we establish the existence of the continuous-time limit process, and we give its representation as a Brownian diffusion. In section 5, we show that the equilibrium strategies of P1 in this class are based on a time-homogeneous Markov chain. Therefore, it is natural to conjecture that the associated equilibrium price processes converge to some time-homogeneous Brownian diffusion when n goes to infinity. More precisely :

CONJECTURE 1: *If μ is the law of Π_1 where Π is a time homogeneous Brownian martingale defined by :*

$$\Pi_0 = \int x d\mu, \qquad \Pi_t = \int_0^t a(\Pi_s) dB_s$$

then for all n there exists an equilibrium in $G_n(\mu)$ such that the process Π^n_t of posted prices at equilibrium converges in law as $n \to \infty$ to Π.

As mentioned in the introduction, all the results in this work are obtained through an auxiliary game, the dual game. The main difficulty is the regularity of the correspondance between equilibria in the dual game and in the initial game. Due to this technical problem, we will only prove in this work the following weaker version of this conjecture under some regularity and ellipticity assumptions on a denoted (A)(see proposition 6.1 for a precise definition). Our result is divided in two parts, the first one is an approximate version of the conjecture and the second is an existence result of equilibrium for fixed n.

THEOREM 2.1: *If μ is the law of Π_1 where Π is a time homogeneous Brownian martingale defined by*

$$\Pi_0 = \int x d\mu, \qquad \Pi_t = \int_0^t a(\Pi_s) dB_s$$

where the function a fulfills assumptions (A), then for all n there exists a probability μ_n and an equilibrium in $G_n(\mu_n)$ such that the process $\Pi^n_t = p^n_{\lfloor nt \rfloor}$ of posted prices at equilibrium converges in law as $n \to \infty$ to Π and $\mu_n \to \mu$.

THEOREM 2.2: *For all non-atomic $\mu \in \Delta(\mathbb{R})$ with compact convex support $[a, b]$ there exists an equilibrium in $G_n(\mu)$ in the class or reduced and self-equalizing equilibria.*

In view of these results, the missing argument to prove the conjecture would be a some local regularity with respect to n and μ of the map (or of a selection of the correspondance) associating to μ the law of the induced process of equilibrium prices.

Note that when μ is a normal or a log-normal distribution, which corresponds to $a(x) = 1$ or $a(x) = x$, we recover the classical dynamics of Bachelier and Black and Scholes.
The conjecture as well as the theorem suppose that the distribution μ is the law at time $t = 1$ of a time-homogeneous Brownian diffusion. This assumption excludes for example to consider dicrete distributions. The behavior of the price process for discrete distributions is still an open question. However, it is not difficult to show, for some examples of discrete distributions μ, that there is no equilibrium in the class of reduced and self-equalizing equilibria that we consider in this work.

Behavioral Strategies in $G_n(\mu)$. We will assume that μ has compact support, the set of such probabilities will be denoted hereafter $\Delta_c(\mathbb{R})$. All the results of the next sections can be extended to a probability with finite first order moment. This extension is only technical and not necessary for the proof of the main theorem. Since the compact support case contains all the relevant ideas of the general proof, we think that this assumption will clarify the presentation. A note on the extension is given in section 8.

At round k, the choice made by P2 is based on his past observations $(p_i, q_i)_{i\leq k-1} \in (\mathbb{R}^2)^{k-1}$. A behavioral strategy τ for P2 in $\Gamma_n(\mu)$ is then a sequence $(\tau_1, ..., \tau_n)$ of transition probabilities

$$\tau_k : (\mathbb{R}^2)^{k-1} \to \Delta(\mathbb{R}).$$

P1 can also use his private information to make his choice, so the price he posts at round k depends on : $(L, (p_i, q_i)_{i\leq k-1}) \in \mathbb{R} \times (\mathbb{R}^2)^{k-1}$. A behavioral strategy for P1 is then a sequence $\sigma = (\sigma_1, ..., \sigma_n)$ of transition probabilities

$$\sigma_k : \mathbb{R} \times (\mathbb{R}^2)^{k-1} \to \Delta(\mathbb{R}).$$

The triplet (μ, σ, τ) induces a unique probability $\Pi_{(\mu,\sigma,\tau)}$ on $\mathbb{R}\times(\mathbb{R}^2)^n$. The payoff functions are :

$$g_1^n(\mu, \sigma, \tau) = E_{\Pi_{(\mu,\sigma,\tau)}}[Ly_n^R + y_n^N] = E_{\Pi_{(\mu,\sigma,\tau)}}[\sum_{k=1}^{n}(L - p_k)sg(p_k - q_k)]$$
$$g_2^n(\mu, \sigma, \tau) = E_{\Pi_{(\mu,\sigma,\tau)}}[Lz_n^R + z_n^N] = E_{\Pi_{(\mu,\sigma,\tau)}}[\sum_{k=1}^{n}(q_k - L)sg(p_k - q_k)]$$

Since these expectations are not always well-defined for integrability reasons, we introduce the notion of admissible strategies. An strategy for P1 is admissible if there exists $M > 0$ such that [1]:

$$\forall k, \forall (L, (p_i, q_i)_{i=1,..,k-1}) \in \mathbb{R} \times (\mathbb{R}^2)^{k-1}, \sigma_k(L, (p_i, q_i)_{i=1,..,k-1}) \in \Delta([-M, M])$$

We denote Σ_n the set of admissible strategies in $G_n(\mu)$. The set $\mathcal{T}_n$ of admissible strategies of P2 is defined similarly. With admissible strategies, the payoff functions are always well-defined in $\mathbb{R}$. A (Nash) equilibrium in $G_n(\mu)$ is then a pair of admissible strategies (σ, τ) such that:

$$\forall \tilde{\sigma} \in \Sigma_n, \quad g_1^n(\mu, \sigma, \tau) \geq g_1^n(\mu, \tilde{\sigma}, \tau)$$
$$\forall \tilde{\tau} \in \mathcal{T}_n, \quad g_2^n(\mu, \sigma, \tau) \geq g_2^n(\mu, \sigma, \tilde{\tau})$$

Reduced strategies in $G_n(\mu)$. As in zero-sum games, we introduce the notion of reduced strategies : reduced strategies are strategies that do not depend on P2's past actions.

A reduced strategy τ for P2 is a sequence $(\tau_1, ..., \tau_n)$ of transition probabilities

$$\tau_k : (\mathbb{R})^{k-1} \to \Delta(\mathbb{R}).$$

A reduced strategy for P1 is a sequence $\sigma = (\sigma_1, ..., \sigma_n)$ of transition probabilities

$$\sigma_k : \mathbb{R} \times (\mathbb{R})^{k-1} \to \Delta(\mathbb{R}).$$

1. In this expression, $\Delta([-M, M])$ is as usual identified with the subset of probabilities on $\mathbb{R}$ supported by $[-M, M]$. Let us also mention that restricting the strategies to be concentrated on a fixed convex compact set containing the support of the law μ would not modify our results.

The main property of the game in reduced strategies is that the action of P2 at step k does not influence the payoff of next stages. Therefore, to play a best reply against some reduced strategy σ of P1, P2 has just to maximize his stage payoff given past actions of P1. In other words, P2 can be seen as a succession of players facing P1, being informed only of the past actions of P1. We will focus in this work on equilibria in reduced strategies. We exclude therefore equilibria where P2 has to follow an equilibrium path and is punished if a deviation is observed. The next proposition shows that an equilibrium in the game restricted to reduced strategies is still an equilibrium in the initial game.

PROPOSITION 2.1: *An equilibrium in the game where the strategy sets of the players are restricted to reduced strategies is an equilibrium in the initial game.*

PROOF. This proof is a direct adaptation of the one appearing in [**23**]. Suppose that (σ^*, τ^*) is an equilibrium in the game $G_n(\mu)$ where players are constrained to use only reduced strategies. For any $\tau \in \mathcal{T}_n$, there exists a reduced strategy $\hat{\tau}$ giving the same payoff as τ against any reduced strategy of P1. The strategy $\hat{\tau}$ proceeds as follows : at step k, P2 does not remind his past actions $(q_1, .., q_{k-1})$, but using past actions of P1, he generates a virtual history $(p_i, \hat{q}_i)_{i=1,..,k-1}$ by choosing $\hat{q}_i$ with the probability $\tau(p_1, \hat{q}_1, .., p_{i-1}, \hat{q}_{i-1})$. He selects then at stage k a price q_k with the probability $\tau((p_i, \hat{q}_i)_{i=1,..,k-1})$. Since the action of P1 does not depend on past actions of P2, the conditional distribution of (p_k, q_k) given $(L, p_1, .., p_{k-1})$ is the same as if P2 was using τ, and so is the conditional expected payoff at stage k. The situation is not symmetric for P1, because to generate a virtual history of the past actions of P2, he has to know which strategy P2 is using. However, the same argument shows that for all $\sigma \in \Sigma_n$, there exists a reduced strategy $\hat{\sigma}$ giving the same payoff as σ against the fixed strategy τ^*, and we have :

$$\forall \sigma \in \Sigma_n,\ g_n^1(\sigma, \tau^*) = g_n^1(\hat{\sigma}, \tau^*) \leq g_n^1(\sigma^*, \tau^*)$$
$$\forall \tau \in \mathcal{T}_n,\ g_n^2(\sigma^*, \tau) = g_n^2(\sigma^*, \hat{\tau}) \leq g_n^2(\sigma^*, \tau^*)$$

□

3. The dual game $G_n^*(\psi)$

The game we introduce now is a generalization of the dual game introduced in [**23**] to analyze the behavior of the uninformed player in zero-sum games with one-sided information.

Representation of P1's reduced strategies. Given a reduced strategy σ of Player 1 in $G_n(\mu)$, the pair (μ, σ) induces a distribution π on $(L, p_1, ..., p_n)$ such that the marginal distribution of L, denoted π_L, is equal to μ. Conversely, any such distribution can be disintegrated in a pair (μ, σ) where σ is the sequence of conditional distributions of p_k given $(L, p_1, .., p_{k-1})$. Since the payoff function in $G_n(\mu)$ depends on σ only through π, we can identify the set of reduced strategies of P1 to the set of distributions π such that $\pi_L = \mu$.

Best reply of P1. Given a reduced strategy τ of P2, the problem P1 is facing when computing a best reply to τ is then (with obvious notations):

$$\sup_{\pi:\pi_L=\mu} g_n(\pi,\tau)$$

It is then natural to relax the constraint introducing Lagrange multipliers. To do this, we can use the linear functionals acting on π_L given by $\pi_L \to \langle \psi, \pi_L\rangle = \int \psi d\pi_L$ with ψ in some (sufficiently large) function space. The above supremum becomes

$$\sup_{\pi} \inf_{\psi} g_n(\pi,\tau) - \langle \psi, \pi_L - \mu\rangle$$

Assuming that the *sup* and *inf* commute in this formula, this leads to the maximization problem

$$\sup_{\pi} g_n(\pi,\tau) - \langle \psi, \pi_L\rangle$$

that can be interpreted as the best reply of P1 in a auxiliary game of complete information called the dual game $G_n^*(\psi)$.

Description of the dual game $G_n^*(\psi)$. At the beginning of this game, P1 selects privately the liquidation value L of the risky asset, and then the game follows as G_n. At the end, a penalty $\psi(L)$ is subtracted from P1's payoff.

More precisely, for a given penalty function ψ, a strategy π for P1 in $G_n^*(\psi)$ is a distribution π on $(L, p_1, .., p_n)$ with compact support or equivalently a pair (μ,σ) where $\mu = \pi_L \in \Delta_c(\mathbb{R})$ and σ is a strategy in Σ_n. P2 chooses a strategy $\tau \in \mathcal{T}_n$. Payoff functions are:

$$g_1^{n*}(\psi,(\mu,\sigma),\tau) = g_1^n(\mu,\sigma,\tau) - \langle \psi, \mu\rangle$$

$$g_2^{n*}(\psi,(\mu,\sigma),\tau) = g_2^n(\mu,\sigma,\tau)$$

We assume that $\psi : \mathbb{R} \to \mathbb{R}\cup\{+\infty\}$ is such that $\langle \psi, \mu\rangle = \mathbb{E}_\mu[\psi(L)]$ is well defined in $\mathbb{R}\cup\{+\infty\}$ for all $\mu \in \Delta_c(\mathbb{R})$ and is not identically equal to $+\infty$. Further assumptions will be made in the next sections. Next lemma shows how the equilibria in the dual game G_n^* are linked with equilibria in the primal game G_n.

LEMMA 3.1: *If $((\mu,\sigma),\tau)$ is an equilibrium in the dual game $G_n^*(\psi)$, then (σ,τ) is an equilibrium in the game $G_n(\mu)$.*

PROOF. Since (μ,σ,τ) is an equilibrium.

$$\forall (\tilde\mu,\tilde\sigma) \in \Delta^1(\mathbb{R}) \times \Sigma_n, \quad g_1^n(\mu,\sigma,\tau) - \mathbb{E}_\mu[\psi(L)] \geq g_1^n(\tilde\mu,\tilde\sigma,\tau) - \mathbb{E}_{\tilde\mu}[\psi(L)]$$

At first, it implies $\mathbb{E}_\mu[\psi(L)] < \infty$, otherwise P1 could deviate by choosing a distribution $\tilde\mu$ such that $\mathbb{E}_{\tilde\mu}[\psi(L)] < \infty$, getting this way a better payoff. Then, since $\mathbb{E}_\mu[\psi(L)]$ does not depend on σ, we have

$$\forall \tilde\sigma \in \Sigma_n, \quad g_1^n(\mu,\sigma,\tau) \geq g_1^n(\mu,\tilde\sigma,\tau)$$

The conclusion follows since the equilibrium condition for τ:

$$\forall \tilde\tau \in \mathcal{T}_n, \quad g_2^n(\mu,\sigma,\tau) \geq g_2^n(\mu,\sigma,\tilde\tau)$$

is the same in $G_n(\mu)$ and in $G_n^*(\psi)$. $\square$

4. Equilibrium Strategy for P2 in the dual game

Representation of P2's reduced strategies. For any vector $(p_1, .., p_n) \in \mathbb{R}^n$, we define $p_{<k} = (p_1, .., p_{k-1})$ and $p_{\leq k} = (p_1, .., p_k)$. A reduced strategy τ is a sequence $(\tau_1, ..., \tau_n)$, where $\tau_k(p_{<k})$ is the lottery used by P2 to select his price at step k given the sequence $p_{<k}$ of past actions of P1. Since any random variable can be represented as a nondecreasing function of an uniform random variable, there exists a measurable function $f_k : \mathbb{R}^{k-1} \times [-1, 1] \to \mathbb{R}$, right-continuous and nondecreasing with respect to the last variable, such that $f_k(p_{<k}, U_k)$ follows the distribution $\tau_k(p_{<k})$ and U_k is a random variable uniformly distributed on $[-1, 1]$. Applying this process inductively, playing τ is equivalent to play $q_k = f_k(p_{<k}, U_k)$ where $(U_1, .., U_n)$ is a sequence of independent random variable uniformly distributed on $[-1, 1]$.

Best reply of P1 in the dual game. Using this representation, if P2 is using the strategy $\tau \cong (f_1, .., f_n)$, then the payoff of P1 if he plays the pure reduced strategy $(L, p_1, .., p_n)$ is :

$$\mathbb{E}[\sum_{k=1}^{n}(L - p_k)sg\,(p_k - f_k(p_{<k}, U_k))] - \psi(L)$$

where the expectation is taken with respect to the variables $(U_1, .., U_n)$. To compute a best reply P1 has to solve the problem :

$$\sup_{L,p_1,..,p_n} \mathbb{E}[\sum_{k=1}^{n}(L - p_k)sg\,(p_k - f_k(p_{<k}, U_k))] - \psi(L)$$

We define

$$\begin{cases} \alpha_k(p_{\leq k}) = \mathbb{E}[sg\,(p_k - f_k(p_{<k}, U_k))] \\ S_k(p_{\leq k}) = \sum_{i=1}^{k} \alpha_i(p_{\leq i}) \end{cases} \tag{4.1}$$

Note that these two functions depend on the strategy of P2. The above quantity becomes

$$LS_n(p_{\leq n}) - \psi(L) - \sum_{k=1}^{n} p_k \alpha_k(p_{\leq k})$$

If we take the supremum over L, we obtain :

$$H(p_{\leq n}) \triangleq \psi^*(S_n(p_{\leq n})) - \sum_{k=1}^{n} p_k \alpha_k(p_{\leq k})$$

where ψ^* is the Fenchel transform of ψ defined by :

$$\forall x \in \mathbb{R}, \quad \psi^*(x) = \sup_{L \in \mathbb{R}} xL - \psi(L).$$

Using Fenchel's lemma, the supremum is reached for $L \in \partial\psi^*(S_n(p_{\leq n}))$ if this subdifferential is nonempty. In particular, if ψ^* admits a derivative in $S_n(p_{\leq n})$, then this relation becomes $L = \psi^{*\prime}(S_n(p_{\leq n}))$.

Self-equalizing strategies. At equilibrium in $G_n^*(\psi)$, P1 will play only optimal values of $p_{\leq n}$ (i.e. maximizing H), and therefore the equilibrium strategies of P2 must be equalizing H on the support of $p_{\leq n}$. Since at round k, P2 posts a price $q_k = f_k(p_{<k}, U_k)$ where U_k is uniformly distributed on $[-1,1]$, $f_k(p_{<k}, [-1,1])$ is the support of the mixed strategy of P2 at round k. Intuitively, P1 has no incentive to play outside this support. Indeed, if he posts a price p_k which is greater than $f_k(p_{<k}, 1)$, then $p_k > q_k$ and P1 will always buy one unit of the risky asset at price p_k. He gets clearly a better payoff by posting a price closer to $f_k(p_{<k}, 1)$. This argument shows that a best reply of P1, if it exists, must be concentrated on the support [2] of the mixed strategy of P2 at round k, and this leads us to introduce a particular class of equalizing strategies. A strategy of P2 is self equalizing if it makes P1 indifferent between any sequence of prices in the support of the mixed action of P2 at every round k. Precisely :

Definition 4.1: *A reduced strategy $(f_1, .., f_n)$ of P2 is self equalizing if there exists some real constant C such that :*

$$H(p_{\leq n}) = C \text{ for all } p_{\leq n} \text{ such that } \forall k \leq n,\ p_k \in f_k(p_{<k}, [-1,1]).$$

The self-equalizing strategies of P2 depend on the function ψ only through its Fenchel transform ψ^*, and we will therefore limit the study of the dual game to proper l.s.c. convex functions ψ. To simplify the notations, we take the convention $\varphi = \psi^*$ and we will index equilibrium strategies in $G_n^*(\psi)$ by φ.

Definition 4.2: *Let D_n denote the set of convex functions f, C^1 on $[-n,n]$ and such that f' is strictly increasing.*

Proposition 4.1: *Assume that $\varphi \in D_n$, then there exists a self-equalizing strategy $\tau(\varphi)$ of P2 in $G_n^*(\psi)$.*
This strategy is defined by $\forall k \leq n,\ f_k(p_{<k}, u) = h(S_{k-1}, u)$, with

$$h(S_{n-1}, u) \triangleq \begin{cases} \frac{1}{u}\left(\varphi(S_{n-1}+u) - \varphi(S_{n-1})\right) & \text{if } u \neq 0 \\ \varphi'(S_{n-1}) & \text{if } u = 0 \end{cases}$$

Moreover, any best reply for P1 to this strategy is a distribution π on $(L, p_1, .., p_n)$ such that with probability 1 :

$$\forall k \leq n,\ p_k \in h(S_{k-1}, [-1,1]) \quad \text{and} \quad L = \varphi'(S_n).$$

Proof. We construct the strategy $\tau(\varphi)$ for P2 in $G_n^*(\psi)$ by backward induction. (We omit the dependence of α_k and S_k on p to shorten notations.)

Let $p_n = f_n(p_{<n}, v_n)$ for some $v_n \in [-1,1]$, then

$$H(p_{\leq n}) = \varphi(S_{n-1}+\alpha_n) - \alpha_n f_n(p_{<n}, v_n) - \sum_{k=1}^{n-1} p_k \alpha_k$$

Suppose that $f_n(p_{<n}, \bullet)$ is strictly increasing, then

$$\alpha_n(p_{\leq n}) = \mathbb{E}[sg\left(f_n(p_{<n}, v_n) - f_n(p_{<n}, U_n)\right)] = \mathbb{E}[sg(v_n - U_n)] = v_n \tag{4.2}$$

2. To be rigorous, the support of q_k is the closure of $f_k(p_{<k}, [-1,1])$ and our argument implies that any best reply of P1 must be concentrated on the convex hull of this support at step k.

The equalizing condition becomes

$$\forall v_n \in [-1,1], \quad \varphi(S_{n-1}+v_n) - v_n f_n(p_{<n}, v_n) - \sum_{k=1}^{n-1} p_k \alpha_k = C$$

Taking $v_n = 0$ implies $C = \varphi(S_{n-1}) - \sum_{k=1}^{n-1} p_k \alpha_k$ and we deduce

$$f_n(p_{<n}, v_n) = h(S_{n-1}, v_n) = \begin{cases} \frac{1}{v_n}\left(\varphi(S_{n-1}+v_n) - \varphi(S_{n-1})\right) & \text{if} \quad v_n \neq 0 \\ \varphi'(S_{n-1}) & \text{if} \quad v_n = 0 \end{cases}$$

The same argument leads by backward induction to:

$$f_k(p_{<k}, v_k) = h(S_{k-1}, v_k)$$

and since h is strictly increasing in its last variable, it shows that the strategy is self-equalizing and that $C = \varphi(0)$.

Note that if at step n, P1 plays $p_n > h(S_{n-1}, 1)$ then he gets strictly less than playing $h(S_{n-1}, 1)$. Indeed $\alpha_n = 1$ and we have:

$$\begin{aligned} H(p_{\leq n}) = & \varphi(S_{n-1}+1) - p_n - \sum_{k=1}^{n-1} p_k \alpha_k \\ & < \varphi(S_{n-1}+1) - h(S_{n-1}, 1) - \sum_{k=1}^{n-1} p_k \alpha_k = \varphi(S_{n-1}) - \sum_{k=1}^{n-1} p_k \alpha_k \end{aligned}$$

If $p_n < h(S_{n-1}, -1)$, then $\alpha_n = -1$ and we have the same inequality. It implies by induction that $H(p_{\leq n}) < \varphi(0)$ if P1 plays at some round k outside of $[h(S_{k-1}, -1), h(S_{k-1}, 1)]$.

A best reply for P1 to this strategy is then any distribution π on $(L, p_1, .., p_n)$ such that with probability 1 : $\forall k \leq n$, $p_k \in h(S_{k-1}, [-1,1])$ and $L = \varphi'(S_n)$.

□

The strategy $\tau(\varphi)$ constructed in the above lemma can be summarized as follows:
- At step 1, P2 plays $q_1 = h(0, U_1)$
- At step k, he computes

$$S_{k-1} = \begin{cases} -1 & \text{if } p_k < h(S_{k-2}, -1) \\ \text{solution of } p_k = h(S_{k-2}, S_{k-1} - S_{k-2}) & \text{if } p_k \in h(S_{k-2}, [-1,1]) \\ 1 & \text{if } p_k > h(S_{k-2}, 1) \end{cases}$$

and plays $q_k = h(S_{k-1}, U_k)$ where $(U_1, .., U_n)$ is a sequence of independent random variables uniformly distributed on $[-1, 1]$.

5. Equilibrium strategy for P1

This section is devoted to prove the following result:

PROPOSITION 5.1: *For all $\varphi \in D_n$, there exists a unique reduced strategy $\pi(\varphi)$ of P1 in $G_n^*(\psi)$ such that $(\pi(\varphi), \tau(\varphi))$ is an equilibrium.*

Since we consider reduced strategies of P1, actions of P2 at round k do not influence P1's behavior during the following rounds. To play a best reply to a reduced strategy, P2 has therefore to maximize at each step his stage payoff conditionally to the past actions of P1. This conditional payoff at step k if P2 plays some strategy τ against the reduced strategy π is:

$$\begin{aligned}&\mathbb{E}[(q_k - L)sg(p_k - q_k) \mid p_{<k}] \\ &\quad = \mathbb{E}[\mathbb{E}[(q_k - L)sg(p_k - q_k) \mid p_{\leq k}] \mid p_{<k}]\end{aligned}$$

Since p_k is a function of $p_{\leq k}$ and by construction L and q_k are conditionally independent given $p_{\leq k}$, we have

$$\mathbb{E}[Lsg(p_k - q_k) \mid p_{\leq k}] = \mathbb{E}[L_k sg(p_k - q_k) \mid p_{\leq k}]$$

where $L_k = \mathbb{E}[L \mid p_{\leq k}]$.

Let $\phi \in D_n$, and define the functions h, S_k, and α_k associated to the strategy $\tau(\varphi)$ as in the previous section. Suppose that $\pi(\varphi)$ is a best reply to $\tau(\varphi)$, hence a distribution on $(L, p_1, .., p_n)$ such that with probability 1:

$$\forall k \leq n,\ p_k \in h(S_{k-1}, [-1,1]) \quad \text{and} \quad L = \varphi'(S_n)$$

Conditions on φ implies that the function $h(s, \bullet)$ is strictly increasing. Therefore such a strategy of P1 is completely determined by the law of the process $(S_k)_{k=1,..,n}$. Indeed, using (4.1),(4.2) and the definition of $\tau(\varphi)$, we find :

$$p_k = h(S_{k-1}, \alpha_k) \quad \text{where} \quad \alpha_k = S_k - S_{k-1}.$$

The random variables $p_{\leq k}$ and $S_{\leq k}$ generate the same σ-field, and we can replace the conditional expectations with respect to "p" by conditional expectations with respect to "S". In particular, $L_k = \mathbb{E}[L \mid S_{\leq k}]$. Suppose now that $(\pi(\varphi), \tau(\varphi))$ is an equilibrium. The strategy $\tau(\varphi)$ is then a best reply to $\pi(\varphi)$, and we will show that this condition determines uniquely the distribution of the process $(S_k)_{k=1,..,n}$ induced by $\pi(\varphi)$. The self-equalizing strategy of P2 $\tau(\varphi)$ being a best reply to $\pi(\varphi)$, it implies that for all k, P2's mixed action is concentrated on the optimal values of q_k. Since $q_k = h(S_{k-1}, U)$, where U is uniformly distributed on $[-1,1]$, the strategy of P1 must be equalizing on the support of q_k, or equivalently there exists a constant C_k independent of t such that for Lebesgue-almost every t in $[-1,1]$:

$$\mathbb{E}_{\pi(\varphi)}[(h(S_{k-1}, t) - L_k)sg(p_k - h(S_{k-1}, t)) \mid S_{<k}] = C_k \tag{5.1}$$

Moreover, since these values of q_k must be optimal, we also have for all $q \in \mathbb{R}$,

$$\mathbb{E}_{\pi(\varphi)}[(q - L_k)sg(p_k - q)) \mid S_{<k}] \leq C_k \tag{5.2}$$

The conditional expectation L_k can be expressed as a function of $S_{\leq k}$. By abuse of notations, we will denote this function by $L_k(S_{\leq k})$. In the same way, C_k is a function of $S_{<k}$ that will be denoted $C_k(S_{<k})$. Using that h is strictly increasing in its last variable and that $p_k = h(S_{k-1}, \alpha_k)$ for $\alpha_k \in [-1,1]$, the equalizing condition (5.1) becomes

$$\mathbb{E}_{\pi(\varphi)}[(h(S_{k-1}, t) - L_k(S_{<k}, S_{k-1} + \alpha_k))sg(\alpha_k - t) \mid S_{<k}] = C_k$$

If $G^k_{s_{<k}}(t)$ denotes the distribution function of α_k conditionally on $S_{<k} = s_{<k}$, then this equation becomes a Stieltjes integral equation for $G^k_{s_{<k}}(t)$:

$$h(s_{k-1},t)(1-2G^k_{s_{<k}}(t)+\Delta G^k_{s_{<k}}(t)) - \int_{[-1,1]} L_k(s_{<k}, s_{k-1}+u)sg(u-t)dG^k_{s_{<k}}(u) = C_k(s_{<k}) \tag{5.3}$$

where $\Delta G^k_{s_{<k}}(t)$ is the jump of the distribution function at time t.

The equalizing condition (5.1) as well as (5.2) need only to hold almost surely with respect to the distribution of $S_{<k}$. We will show that the equation (5.3) for any fixed value $s_{<k}$ has only one solution $G^k_{s_{<k}}$. Therefore, (5.1) implies that $G^k_{S_{<k}}$ is a version of the conditional distribution of α_k given $S_{<k}$, and thus the law of the the process $(S_k)_{k=1,..,n}$ is completely described by the functions $(G^k_{s_{<k}})_{k=1,..,n}$. The proof is based on three technical lemma proved in section 8.

Lemma 5.1: *Assuming* (5.2) *and that equation* (5.3) *holds for Lebesgue almost-every t in* $[-1,1]$ *then* (5.3) *holds for all t in* $[1,1]$, *any solution $G^k_{s_{<k}}(t)$ is continuous in $t=-1$ and* 1 *and*

$$C_k(s_{<k}) = \frac{1}{2}(h(s_{k-1},-1)-h(s_{k-1},1)) \triangleq C(s_{k-1})$$

$$L_{k-1}(s_{<k}) = \frac{1}{2}(\varphi(s_{k-1}+1)-\varphi(s_{k-1}-1)) \triangleq g(s_{k-1})$$

The last assertions in the previous lemma imply that equation (5.3) for $k<n$ depends only on $s_{<k}$ through s_{k-1}. Lemma 5.3 will show that this equation has only one solution, and thus the process $(S_k)_{k=1,..,n-1}$ associated to these solutions will be a time-homogeneous Markov chain. Let us first study the case $k=n$ in the next lemma.

Lemma 5.2: *Using the boundary condition $G^n_{s_{<n}}(-1)=0$ and $G^n_{s_{<n}}(1)=1$, the unique solution of* (5.3) *for $k=n$ is $G^n_{s_{<n}}(t)=\frac{1}{2}(1+t)$.*

Equalizing condition for P1 implies therefore that $p_n = h(S_{n-1},U)$ where U is uniformly distributed on $[-1,1]$ and independent of S_{n-1}. Taking the derivative with respect to t in (5.3) for $k<n$, we obtain

$$\frac{\partial h}{\partial t}(s_{k-1},t)(1-2G^k_{s_{<k}}(t)) + 2G^k_{s_{<k}}{}'(t)(g(s_{k-1}+t)-h(s_{k-1},t)) = 0 \tag{5.4}$$

Lemma 5.3: *For $k<n$, equation* (5.3) *admits a unique solution depending only on s_{k-1} and denoted $G_{s_{k-1}}$. This solution is piecewise C^1 and solution of* (5.4) *except in finitely many points in $(-1,1)$. Moreover, if we assume that φ is C^2 and that $\varphi''>0$, then $G_{s_{k-1}}$ is continuous.*

The equalizing strategy of P1 is then completely described by :

$$\forall k \le n,\, p_k = h(S_{k-1}, S_k - S_{k-1}), \quad L = \varphi'(S_n)$$

where $S_1,...,S_{n-1}$ is a time-homogeneous markov chain such that:

$$\mathbb{P}[S_k - S_{k-1} \le t \mid S_{k-1}=s] = G_s(t)$$

with $G_s(t)$ the solution given by lemma 5.3 (see section 8 or 7 for a precise formula) and $S_n = S_{n-1}+U$ with U uniformly distributed on $[-1,1]$ and independent of S_{n-1}. To conclude that $(\pi(\varphi),\tau(\psi))$ is an equilibrium, since P1's strategy is reduced, we only need to check condition (5.2), which says that P2 cannot obtain a better payoff at step k by playing outside of $h(S_{k-1}, [-1, 1])$. Indeed, if at round k P2 plays $q_k > h(S_{k-1}, 1)$, then his conditional stage payoff given $S_{<k}$ is:

$$\mathbb{E}_{\pi(\varphi)}[(q_k - g(S_{k-1} + \alpha_k))(-1) \mid S_{<k}] = -q_k + g(S_{k-1}) < -h(S_{k-1}, 1) + g(S_{k-1}) = C(S_{k-1})$$

and the same argument gives the corresponding inequality in the symmetric case, which concludes the proof.

6. Asymptotic behavior of equilibria in the dual game

We now analyze the behavior of the equilibria constructed in the previous section when n goes to infinity. In the zero-sum setting, the main result used to obtain the limit price process is the asymptotic expansion of the value of zero-sum games. In the game studied in [**31**] or more generally in the class of games introduced in [**25**], using a central limit result, the first term of this expansion is shown to be or order $\sqrt{n}$. Since the game we consider here is close to the zero-sum market game introduced in [**23**], it is then natural to consider the games where the payoffs functions are divided by $\sqrt{n}$ and to study the asymptotic behavior of the equilibria obtained in the previous section in these game. Consider the games $G_n(\mu)$ where the payoff functions are divided by $\sqrt{n}$. In particular, the payoff function of P1 is then:

$$\frac{1}{\sqrt{n}} g_1^n(\mu, \sigma, \tau)$$

Introducing the dual games associated to this game in the same way as for G_n, the payoff function of P1 is then :

$$\frac{1}{\sqrt{n}} g_1^n(\mu, \sigma, \tau) - \langle \psi, \mu \rangle$$

In order to analyze the asymptotic behavior of these games for a fixed function ψ, it is then equivalent to study the sequence of dual games $G_n^*(\psi_n)$ with $\psi_n = \sqrt{n}\psi$. In this case $\varphi_n(y) = \psi_n^*(y) = \sqrt{n}\varphi(\frac{y}{\sqrt{n}})$. With these notations, we are now able to state the main convergence result on the reduced and self-equalizing equilibria in the dual game.

Proposition 6.1: *Let φ be a convex function, C^4 on $\mathbb{R}$ such that ϕ' is bounded, $\varphi'' > 0$ and*

$$\frac{\left|\varphi^{(3)}(z)\right|}{6\varphi''(z)} \leq C(1 + |z|) \tag{6.1}$$

for some constant C. Let $(p_k^n)_{k=1,..,n}$ denote the process of prices posted by P1 when he's playing the equilibrium strategy constructed in proposition 5.1 in $G_n(\psi_n)$, and define the continuous-time process:

$$\Pi_t^n = p_{\lfloor nt \rfloor}^n \text{ for } t \in [0, 1]$$

With these assumptions, the sequence of processes Π^n converge in distribution, in the space of cádlág functions defined on $[0, 1]$ endowed with the Skorokhod topology, to a continuous diffusion process defined as the unique solution (in law) of the stochastic differential equation :

$$\Pi_t = \varphi'(0) + \int_0^t \frac{1}{\sqrt{3}\psi''(\Pi_s)} dB_s$$

where B is a standard Brownian motion.

PROOF. In the dual game $G_n^*(\psi_n)$, the equalizing strategy of P1 is described by the process S^n. Let $Z^n = \frac{S^n}{\sqrt{n}}$. Then for all $i < n-1$, the conditional distribution function of $S_{i+1}^n - S_i^n$ given Z_i^n is $G_{\frac{1}{\sqrt{n}}, Z_i^n}$ where $G_{\epsilon,z}$ is solution of

$$(6.2) \quad 2G'_{\epsilon,z}(t)\left(\frac{\varphi(z+\epsilon(t+1)) - \varphi(z+\epsilon(t-1))}{2\epsilon} - \frac{\varphi(z+\epsilon t) - \varphi(z)}{\epsilon t}\right) = (2G_{\epsilon,z}(t)-1)\frac{1}{t}\left(\varphi'(z+\epsilon t) - \frac{\varphi(z+\epsilon t)-\varphi(z)}{\epsilon t}\right)$$

with the conditions $G_{\epsilon,z}(-1) = 0$ and $G_{\epsilon,z}(1) = 1$. The last increment $S_n^n - S_{n-1}^n$ is uniformly distributed on $[-1,1]$ and independent of the past of the process. To study the asymptotic of Z^n, we define the continuous-time process $\tilde{Z}_t^n = Z_{\lfloor nt \rfloor}^n$. In order to apply a general approximation result of diffusion by discrete-time Markov chain, we need to have estimates on the conditional moments per unit of time of the process $\tilde{Z}^n$. Namely :

$$b^n(z) = n\mathbb{E}[Z_{i+1}^n - Z_i^n \mid Z_i^n = z]$$
$$c^n(z) = n\mathbb{E}[(Z_{i+1}^n - Z_i^n)^2 \mid Z_i^n = z]$$
$$d^n(z) = n\mathbb{E}[(Z_{i+1}^n - Z_i^n)^3 \mid Z_i^n = z]$$

First note that $d^n(z) \leq \frac{1}{\sqrt{n}}$ and thus converges to 0 uniformly in z.

Let $I_k(\epsilon,z) = \int_{-1}^1 (2G_{\epsilon,z}(t)-1)t^k dt$.

Integrating by parts (6.2) over $[-1,1]$ gives [3]

$$(6.3) \quad \int_{-1}^1 (2G_{\epsilon,z}(t)-1)\frac{\varphi'(z+\epsilon(t+1)) - \varphi'(z+\epsilon(t-1))}{2}dt = \frac{1}{2\epsilon}(\varphi(z+2\epsilon) - \varphi(z-2\epsilon)) - \frac{1}{\epsilon}(\varphi(z+\epsilon) - \varphi(z-\epsilon))$$

Expanding in power of ϵ, we find

$$I_0(\epsilon,z)\varphi''(z) + \epsilon\varphi^{(3)}(z)(I_1(\epsilon,z)-1) = \epsilon^2 R_1(z,\epsilon)$$

where $R_1(z,\epsilon)$ is the rest appearing in the Taylor expansion, that can be expressed as follows

$$R_1(z,\epsilon) = \frac{1}{24}\Big[8\phi^{(4)}(z+2\theta_1\epsilon) - 8\phi^{(4)}(z-2\theta_2\epsilon) + \phi^{(4)}(z+\theta_3\epsilon) - \phi^{(4)}(z-\theta_4\epsilon) - 2\int_{-1}^1 (2G_{\epsilon,z}(t)-1)\left((1+t)^3\phi^{(4)}(z+(1+t)\epsilon\theta_5(t)) - (-1+t)^3\phi^{(4)}(z+(t-1)\epsilon\theta_6(t))\right)dt\Big]$$

where $\theta_i \in [0,1]$ for $i = 1,..,6$. Therefore, we obtain easily the following bound :

$$|R_1(z,\epsilon)| \leq \frac{41}{12} \underset{x\in[-2\epsilon,2\epsilon]}{sup} \left|\varphi^{(4)}(z+x)\right|.$$

The same process applied to (6.2) multiplied by t leads to:

$$\varphi''(z)(I_1(\epsilon,z) - \frac{2}{3}) = \epsilon R_2(z,\epsilon)$$

3. Integration by parts formula holds here, since with our assumptions on φ, the solution of (6.2) is continuous and piecewise C^1, hence absolutely continuous (see the proof in section 8 for the details).

with $|R_2(z,\epsilon)| \leq \frac{52}{9} \underset{x\in[-2\epsilon,2\epsilon]}{sup} \left|\varphi^{(3)}(z+x)\right|$.
We deduce then, using integration by parts

$$c^n(z) = (1 - I_1(\frac{1}{\sqrt{n}}, z)) = \frac{1}{3} - \frac{1}{\sqrt{n}} \frac{R_2(z, \frac{1}{\sqrt{n}})}{\varphi''(z)}$$

$$b^n(z) = -\frac{\sqrt{n}}{2} I_0(\frac{1}{\sqrt{n}}, z) = -\frac{\varphi^{(3)}(z)}{6\varphi''(z)} + \frac{1}{2\sqrt{n}\varphi''(z)} \left(\frac{\varphi^{(3)}(z) R_2(z, \frac{1}{\sqrt{n}})}{\varphi''(z)} - R_1(z, \frac{1}{\sqrt{n}}) \right)$$

With our assumptions on φ, the functions c^n and b^n converge uniformly on compact sets to $c(z) = \frac{1}{3}$ and $b(z) = -\frac{\varphi^{(3)}(z)}{6\varphi''(z)}$. By Corollary 7.4.2 in [**34**], [4] the sequence of processes $\tilde{Z}^n$ converge in distribution, in the space of cádlág functions defined on $[0,1]$ endowed with the Skorokhod topology, to the law of the process Z defined as the solution [5] of :

$$\begin{cases} Z_t = \int_0^t \frac{1}{\sqrt{3}} dB_s - \int_0^t \frac{\varphi^{(3)}(Z_s)}{6\varphi''(Z_s)} ds \\ Z_0 = 0 \end{cases} \tag{6.4}$$

where B is a standard Brownian motion defined on some filtered probability space.

We can now deduce the asymptotic behavior of P2's expected beliefs martingale L^n and of the price process p^n at equilibrium in $G_n^*(\psi_n)$. According to the previous section we have :

$$\begin{aligned} L_k^n &= \int_{-1}^1 \varphi'(Z_k^n + \frac{t}{\sqrt{n}}) \frac{dt}{2} \\ p_k^n &= \frac{\varphi(Z_k^n) - \varphi(Z_{k-1}^n)}{Z_k^n - Z_{k-1}^n} = \frac{1}{Z_k^n - Z_{k-1}^n} \int_0^{Z_k^n - Z_{k-1}^n} \varphi'(Z_{k-1}^n + t) dt \end{aligned}$$

We define the continuous time processes $\tilde{L}_t^n = L_{\lfloor nt \rfloor}^n$ and $\Pi_t^n = p_{\lfloor nt \rfloor}^n$ for $t \in [0,1]$. The limit process Z having continuous trajectories, using Skorokhod representation theorem, there exists some probability space Ω where are defined a sequence $(\tilde{Z}^n)_{n\geq 0}$ of processes and a process Z having the same distribution as above and such that the sequence of trajectories converge almost surely, uniformly in t :

$$\underset{t\in[0,1]}{sup} \left|\tilde{Z}_t^n - Z_t\right| \underset{n\to\infty}{\longrightarrow} 0 \text{ a.s.} \tag{6.5}$$

To conclude that the two processes $\tilde{L^n}$ and Π^n converge in distribution to $\varphi'(Z)$, we only have to check that their trajectories converges almost surely uniformly in t in Ω. Using (6.5), the trajectories of $\varphi'(\tilde{Z}^n)$ converge uniformly in t to that of $\varphi'(Z)$ with probability 1.

---

4. Our process is not a homogeneous Markov chain because of the last increment. But this process and the Markov chain obtained by replacing this last increment using the same transition probability have the same asymptotic behavior. Indeed, assume that the $\tilde{Z}^n$ and the modified versions $\hat{Z}^n$ are defined on the same probability space, then the difference converges uniformly to 0 with probability 1.

5. The conditions on φ ensures that the solution Z of the above stochastic differential equation exists, is unique in law and, by (6.1), does not explode in a finite time.

The result follows then from the two following inequalities.

$$\begin{aligned}\left|\tilde{L}^n_t - \varphi'(\tilde{Z}^n_t)\right| &\leq \int_{-1}^{1} \left|\varphi'(Z^n_{\lfloor nt\rfloor} + \frac{u}{\sqrt{n}}) - \varphi'(Z^n_{\lfloor nt\rfloor})\right| \frac{du}{2} \\ &\leq \frac{1}{\sqrt{n}} \sup_{|u|\leq \frac{1}{\sqrt{n}}} \left|\varphi''(Z^n_{\lfloor nt\rfloor} + u)\right| \\ \left|\Pi^n_t - \varphi'(\tilde{Z}^n_t)\right| &\leq \frac{1}{\sqrt{n}} \sup_{|u|\leq \frac{1}{\sqrt{n}}} \left|\varphi''(Z^n_{\lfloor nt\rfloor} + u)\right|\end{aligned}$$

Using Ito's formula :

$$\varphi'(Z_t) = \varphi'(0) + \int_0^t \frac{1}{\sqrt{3}}\varphi''(Z_s)dB_s$$

Let $\Pi_t = \varphi'(Z_t)$, then by Fenchel lemma $Z_t = \psi'(\Pi_t)$ and

$$\varphi''(Z_s) = \varphi''(\psi'(\Pi_s)) = \frac{1}{\psi''(\Pi_s)}$$

We can now conclude that

$$\Pi_t = \varphi'(0) + \int_0^t \frac{1}{\sqrt{3}\psi''(\Pi_s)} dB_s$$

Since ϕ' is bounded Π is a well-defined martingale and uniqueness in law for the above equation follows from uniqueness in (6.4). □

The set of assumptions on the function a announced in the introduction is just the reformulation of the assumptions made on ϕ in the preceding proposition.

Definition 6.1: *The function a fulfills assumptions (A) if there exists a function ϕ fulfilling the assumptions of proposition 6.1 such that $a(x) = \frac{1}{\sqrt{3}\psi''(x)}$ for x in the range of ϕ' and zero elsewhere.*

Now the main theorem is just a corollary of the preceding proposition.

Proof of theorem 2.1. Define the function $a(x) = \frac{1}{\sqrt{3}\psi''(x)}$ with ψ, φ as in the previous proposition 6.1. The probability μ is then the law of Π_1. Using the notations of the previous proof, the sequence of equilibria in $G^*_n(\psi_n)$ induces a sequence of equilibria in $G_n(\mu_n)$ where μ_n is the law of $\varphi'_n(S_n) = L^n_n$. Since $\tilde{L}^n_t$ converges in distribution to the continuous process Π_t, we deduce that $\tilde{L}^n_1$ converges in distribution to Π_1, hence $\mu_n \to \mu$. The equilibrium price process in $G_n(\mu_n)$ being the same as in $G^*_n(\psi_n)$, the result follows directly from proposition 6.1. □

7. Existence of equilibria in the primal game

The aim of this section is to prove the existence theorem 2.2, which added to the result of the previous section is a first step toward the conjecture as announced in the introduction.

In section 5, P1's equilibrium strategy in $G^*_n(\psi)$ is constructed using a time-homogeneous Markov transition depending on ψ^*. For $f \in D_n$, this Markov transition P^f from

$[-(n-2),(n-2)]$ to $[-1,1]$ is defined by its distribution function:

$$(7.1)\qquad G^f(s,t) = P^f(s,[-1,t]) = \begin{cases} \frac{1}{2}(1-exp(\int_{-1}^{t}\frac{\frac{1}{x}(f'(s+x)-h^f(s,x))}{g^f(s+x)-h^f(s,x)}dx)) & t\in[-1,\delta^f(s)) \\ \frac{1}{2} & t\in[\delta^f(s),\gamma^f(s)) \\ \frac{1}{2}(1+exp(-\int_{t}^{1}\frac{\frac{1}{x}(f'(s+x)-h^f(s,x))}{g^f(s+x)-h^f(s,x)}dx)) & t\in(\gamma^f(s),1] \end{cases}$$

where:

$$h^f(s,x) = \frac{1}{x}(f(s+x)-f(x))$$
$$g^f(x) = \frac{1}{2}(f(x+1)-f(x-1))$$
$$\delta^f(s) = \inf\{x\in[-1,1] \mid g^f(s+x)-h^f(s,x)=0\}$$
$$\gamma^f(s) = \sup\{x\in[-1,1] \mid g^f(s+x)-h^f(s,x)=0\}$$

As shown in lemma 5.3, $P^f(s,dt)$ is the unique solution $Q\in\Delta([-1,1])$ of:

$$(7.2)\ \forall t\in[-1,1],\quad h^f(s,t)(1-2Q([-1,t])+Q(\{t\}))-\int_{[-1,1]} g^f(s+u)sg(u-t)Q(du) = C^f(s)$$

with $C^f(s) = \frac{1}{2}(h^f(s,-1)-h^f(s,1))$.

Definition 7.1: *Let $\overline{D}_n$ be the set of convex functions, C^1 on $[-n,n]$, endowed with the following topology. A sequence f_k converges to f in $\overline{D}_n$ ($f_k\to f$ hereafter) if :*
f_k and f'_k converge uniformly to f and f' in $[-n,n]$.

Let us extend the former definition to $\overline{D}_n$

Definition 7.2: *$P^f(s,.)$ denotes the set of solutions $Q\in\Delta([-1,1])$ of* (7.2).

Proposition 7.1: *The set-valued mapping*

$$(s,f)\in[-(n-2),(n-2)]\times\overline{D}_n \to P^f(s,.)\subset\Delta([-1,1])$$

has nonempty closed-convex values and its graph is closed. Moreover, any distribution $Q\in P^f(s,.)$ is uniquely determined on the intervals $[-1,\delta^f(s))$ and $(\gamma^f(s),1]$, where its distribution function is given by formula (7.1).

Proof. Convexity of $P^f(s,.)$ follows directly from the linearity of (7.2) with respect to Q. Non-emptiness will follow from the next result, using that D_n is dense in $\overline{D}_n$ for the considered topology. Let $s_k\to s\in[-(n-2),(n-2)]$ and $f_k\to f\in\overline{D}_n$. Let also Q_k be a sequence from $P^{f_k}(s_k,dt)$ converging to some probability Q. We will show that (7.2) is satisfied by Q. Therefore, Q will be in the set of solutions $P^f(s,dt)$ of (7.2), and the closedness of the graph will be proved.
At first, $f_k\to f$ implies that $g^{f_k}(s_k+x)$ and $h^{f_k}(s_k,x)$ converge to $g^f(s+x)$ and $h^f(s,x)$ uniformly in x, and also the convergence of $C^{f_k}(s_k)$ to $C^f(s)$. Suppose that $t\in[-1,1]$ is such that $Q(\{t\})=0$. Then by the classical Portemanteau theorem:

$$Q_k([-1,t])\to Q([-1,t]),\quad Q_k(\{t\})\to 0$$

The sequence of function $g^{f_k}(s_k+u)sg(u-t)$ converges to $g^f(s+u)sg(u-t)$ uniformly in u and the only discontinuity point of these functions is t. Therefore, by lemma 8.1

$$\int_{[-1,1]} g^{f_k}(s_k+u)sg(u-t)Q_k(du) \to \int_{[-1,1]} g^f(s+u)sg(u-t)Q(du)$$

As a consequence, Q satisfies (7.2) for all t in the set $J=\{t\in[-1,1], Q(\{t\})=0\}$. But this set is dense in $[-1,1]$ and therefore the right and left-hand limits in of this equation must be equal for all $t\in(-1,1)$. This implies

$$Q(\{t\})(g^f(s+t)-h(s,t))=0$$

Since this condition is equivalent to the continuity in t of the left-hand side of (7.2), Q satisfies (7.2) for all t in $(-1,1)$. To conclude, we only show that the equation still holds for $t=-1$, the case $t=1$ being symmetric. Using the proof of lemma 5.3, we know that if $\delta^f(s)>-1$, then the probability $Q\in P^f(s,.)$ is uniquely determined on $[-1,\delta^f(s))$ (the same proof works in $\overline{D}_n$). On the other hand, if $\delta^f(s)=-1$, then $h^f(s,-1)=g^f(s-1)$. In both cases, equation (7.2) for $t=-1$ is

$$h^f(s,-1)(1)-\int_{[-1,1]} g^f(s+u)Q(du)=C^f(s)$$

This equation holds by assumption when replacing f by f_k and s by s_k. The result with f and s follows therefore from the above mentioned uniform convergences.

□

Notation 12: *Given $f\in\overline{D}_n$, the class of (laws of) processes $\mathbb{S}(f)$ is defined as follows:*
The law of a process $(S_i^f)_{i=1,..,n}$ belongs to $\mathbb{S}(f)$ if $(S_i^f)_{i=1,..,n-1}$ is a Markov chain whose transition is given by

$$\mathbb{P}(S_k^f-S_{k-1}^f\in B \mid S_{k-1}^f)=Q^f(k,S_{k-1}^f)(B)$$

for all Borel set B where $Q^f(k,s)$ is a measurable selection of $s\to P^f(s,.)$. The last increment is defined by $S_n^f=S_{n-1}^f+U$ where U is a random variable uniformly distributed on $[-1,1]$ and independent of S_{n-1}^f.
The set of laws of $f'(S_n^f)$ when the law of the process S^f is varying in $\mathbb{S}(f)$ is denoted $\mathcal{L}(f)$. Note that both these sets are reduced to a point when $f\in D_n$.

As shown in the previous sections, the transition P^f for $f\in D_n$ induces an equilibrium in $G_n^*(f^*)$ and thus an equilibrium in $G_n(\mathcal{L}(f))$. This result is no more true when f is only assumed to be in $\overline{D}_n$, even for measurable selections of the associated set-valued mappings.

Lemma 7.1: *The set-valued mappings associating to $f\in\overline{D}_n$ the set of laws $\mathbb{S}(f)$ and the set of possible laws $\mathcal{L}(f)$ have closed graphs.*

Proof. Given a sequence f_q converging to f and a sequence of processes $(S_k^{f_q})_{k=1,..,n}$ in $\mathbb{S}(f_q)$ that converges in distribution to $(S_k)_{k=1,..,n}$, we will show that the law of the limit process S belong to $\mathbb{S}(f)$. Let ρ be a bounded continuous function on $[-n,n]$. Using the previous lemma, since $S_1^{f_q}$ converges in distribution to S_1, the law of S_1 belongs to $P^f(0,dt)$. We can assume that all the random variables are defined on the same probability space and that $S_k^{f_q}$

converges almost surely to S_k for all $k = 1, .., n$. Let us denote the support function of $P^f(s, .)$ by $\sigma^f(s, \rho)$ defined for $\rho \in C([-1, 1])$ by

$$\sigma^f(s, \rho) = \underset{\nu \in P^f(s,.)}{sup} \int \rho d\nu$$

In order to prove that the conditional law of $S_2 - S_1$ given S_1 is almost surely in the closed convex set $P^f(S_1, .)$, it is sufficient to show that for all $\rho, \eta \in C([-1, 1])$ with $\eta \geq 0$

$$\mathbb{E}[\rho(S_2 - S_1)\eta(S_1)] \leq \mathbb{E}[\sigma^f(S_1, \rho)\eta(S_1)] \tag{7.3}$$

Indeed, a monotone class argument allows to replace η by any indicator function, implying that the inequality holds for conditional expectations. Then, this inequality holds with probability one for all ρ in a dense subset of $C([-1, 1])$, which shows that the conditional distribution belongs almost surely to $P^f(S_1, .)$. It remains to show (7.3). Note that for all q, by assumption

$$\mathbb{E}[\rho(S_2^{f_q} - S_1^{f_q})\eta(S_1^{f_q})] \leq \mathbb{E}[\sigma^{f_q}(S_1^{f_q}, \rho)\eta(S_1^{f_q})] \tag{7.4}$$

The left-hand side converges to the left-hand side of (7.3) by dominated convergence. For the right-hand side, it is sufficient to show that the map $(s, f) \to \sigma^f(s, \rho)$ is upper-semi-continuous. But this follows directly from the continuity of the application $\nu \to \int \rho d\nu$ on $\Delta([-1, 1])$ and the closed graph property given in the previous lemma. The same argument shows by induction that the law of S belongs to $\mathbb{S}(f)$, using independence for the last increment. For the second assertion, recall that $\mathcal{L}(f_q)$ is the set of laws of $f'_q(S_n^{f_q})$ for some processes S^{f_q} as described above. Suppose that the sequence $f'_q(S_n^{f_q})$ converges in law to μ and that f_q converges to f. Note that the sequence of laws of $S_n^{f_q}$ is relatively compact in the weak topology. Since f'_q converges uniformly to f', using lemma 8.1, convergence in law of some subsequence of $S_n^{f_q}$ to S^n implies the convergence in law along the same subsequence of $f'_q(S_n^{f_q})$ to $f'(S_n)$. Identifying the limit and using the previous result, we deduce that the limiting law $\mu = law(f'(S_n)) \in \mathcal{L}(f)$. $\square$

We show next that the set of distributions $\mathcal{L}(f^*)$ for $f \in D_n$ is dense in $\Delta(\mathbb{R})$. More precisely,

Lemma 7.2: *Given an integer K and a sequence $a_0 < .. < a_K$ of real numbers, then for all $\lambda \in int(\Delta_K)$ where Δ_K is the K-dimensional simplex, there exists a function $f \in D_n$ such that $\mathcal{L}(f)$ is a non-atomic measure and*

$$\forall i = 1, .., n, \quad \mathcal{L}(f)([a_{i-1}, a_i]) = \lambda_i$$

identifying $\mathcal{L}(f)$ to its single element.

Proof. For $\epsilon > 0$ such that $2n - \epsilon K > 0$, we define

$$\Delta_K^\epsilon = \{\lambda \in \Delta_K \mid \min_i \lambda_i \geq \epsilon\}$$

For each $\lambda \in \Delta_K$ we define a function $g_\lambda : [-n, n] \to [a_0, a_K]$ as the piecewise linear function such that $g_\lambda(t_i) = a_i$ for $i = 0, .., K$ where

$$t_0 = -n, \ t_K = n, \text{ and for all } i = 1, .., K \quad t_i - t_{i-1} = \epsilon + (2n - \epsilon K)\lambda_i$$

We define then $f_\lambda(t) = \int_{-n}^{t} g_\lambda(u)du$ and the application $\lambda \to f_\lambda$ is clearly continuous from Δ_K to D_n. Using results and notations of the previous section, the application $\lambda \to \mathcal{L}(f_\lambda)$ is continuous and if $\lambda_i = 0$, we have

$$\begin{aligned}\mathcal{L}(f_\lambda)([a_{i-1},a_i]) &= \mathbb{P}(g_\lambda(S_n^{f_\lambda}) \in [a_{i-1},a_i]) = \mathbb{P}(S_n^{f_\lambda} \in [t_{i-1},t_i]) \\ &= \mathbb{E}(\mathbb{P}(S_n^{f_\lambda} \in [t_{i-1},t_i] \mid S_{n-1}^{f_\lambda})) \\ &\leq \frac{t_i - t_{i-1}}{2} = \frac{\epsilon + \lambda_i(2n-\epsilon K)}{2} \leq \frac{\epsilon}{2}\end{aligned}$$

We now define the application:

$$T : \Delta_K \to \Delta_K : \lambda \to (\mathcal{L}(f_\lambda)([a_{i-1},a_i]))_{i=1,..,K}$$

Since the measures $\mathcal{L}(f)$ for $f \in D_n$ are non-atomic, T is continuous and by the above inequality

$$\lambda_i = 0 \Rightarrow T(\lambda)_i \leq \frac{\epsilon}{2} \tag{7.5}$$

We show next that $\Delta_K^{\epsilon/2} \subset T(\Delta_K)$ and the conclusion follows by letting ϵ go to 0.

Fix some $y \in \Delta_K^{\epsilon/2}$ and define $C_i = \{\lambda \in \Delta_K \mid T(\lambda)_i \leq y_i\}$. Then by (7.5) we have $\{\lambda \in \Delta_K \mid \lambda_i = 0\} \subset C_i$ and $\cup_{i=1}^n C_i = \Delta_K$. The KKM lemma implies that $\cap_{i=1}^n C_i \neq \emptyset$ and this completes the proof.

□

Finally, we are able to state an existence result.

PROPOSITION 7.2: *For all non-atomic $\mu \in \Delta(\mathbb{R})$ with compact convex support $[a,b]$ there exists $f \in D_n$ such that $\mathcal{L}(f) = \mu$.*

PROOF. Let $\mu_k = \mathcal{L}(f_k)$, with f_k a sequence of functions in D_n given by the previous lemma such that $\mu_k \to \mu$. The sequence f_k can be chosen such that $f_k(-n) = 0$ and $f'_k([-n,n]) = [a,b]$. Identifying the sequence of derivatives f'_k with distributions functions, the sequence is relatively compact for the convergence in distribution, and we can extract a subsequence also denoted f'_k converging to some limit f' which is a right-continuous nondecreasing function such that $f'(-n) \geq a$ and $f'(n) = b$. By compactness, we can assume that along the same subsequence, the sequence of laws of $S_{n-1}^{f_k}$ converges in law to some variable Y, implying that $S_n^{f_k}$ converges in law to $Y + U$ where U is uniformly distributed on $[-1,1]$ and independent of Y. Since $Y + U$ is non-atomic, lemma 8.1 implies that μ_k converges to $f'(Y+U) \sim \mu$. Since $[a,b]$ is by assumption the support of μ and using the previously mentioned conditions on μ, f' is necessarily continuous on $[-n,n]$. Using that $f(t) = \int_{-n}^{t} f'(u)du$ is in $\overline{D}_n$, the previous reasoning implies that f_k converges to f in $\overline{D}_n$ and therefore μ is the law of $f'(S_n^f)$ for some process S^f as defined in the previous lemma. It remains to show that f is increasing in order to prove $f \in D_n$. Define now

$$D_r = \{s \in [-(n-2), n-2]\,, \text{f is linear on } [s,s+1]\}$$

$$D_l = \{s \in [-(n-2), n-2]\,, \text{f is linear on } [s-1,s]\}$$

Using the characterization of the solutions $Q \in P^f(s,.)$ on the intervals $[-1, \delta^f(s))$ and $(\gamma^f(s), 1]$, if $s \notin D_r$, then these two intervals are not trivial and

$$\forall Q \in P^f(s,.),\ \forall \varepsilon > 0, \quad Q([1-\varepsilon, 1)) > 0 \tag{7.6}$$

A similar result holds for D_l. Suppose that D_r does not contain any neighborhood of integers of type $[k-\varepsilon, k]$ for $k = 0, .., n-2$ and the symmetric property for D_l. Under this assumption, using property (7.6), and since the last step of the process S^f is an uniform random variable, we deduce easily that S_n^f has no atom and that its support is the whole interval $[-n, n]$. Therefore, f' is necessarily increasing otherwise the law $\mu \sim f'(S_n^f)$ would have atoms, which would contradicts our assumption. If D_r (or D_l) contains some neighborhood as described above, the argument is slightly different. We will prove that S_n^f gives positive probability to the set $Z = (D_r + [0,1]) \cup (D_l + [-1, 0])$. The main idea is that if the process S^f is not in D_l, then the next increment is with positive probability in $(-1, -1+\varepsilon]$ for any small $\varepsilon > 0$. Assume that there is a step k at which the process will reach the set $[k-\varepsilon, k] \subset D_r$ with positive probability. We claim that the process cannot "exit" from Z with probability 1. If $S_k^f \in D_r \cap D_l$, then $S_k^f + [-1, 1] \subset Z$ and this implies $\mathbb{P}(S_{k+1}^f \in Z) > 0$. If $\mathbb{P}(S_k^f \notin D_l, S_k^f \in [k-\varepsilon, k]) > 0$, then by integration $\mathbb{P}(S_{k+1}^f \in (k-1-\varepsilon, k-1+\varepsilon)) > 0$. In this last case, if $S_{k+1} \notin Z$, then the next increment will jump with positive probability in an interval $(k-\varepsilon, k+\varepsilon) \subset Z$. Note that the last argument works even if $k+1 = n-1$ since in this case the last increment is uniform. We deduce by induction that in all cases $\mathbb{P}(S_n^f \in Z) > 0$. But, this implies that S_n^f gives positive probability to some interval on which f is linear, which in turn implies that μ has some atom, and contradicts our assumption. $\square$

8. Technical results.

Proofs of the lemmas in section 5.

Proof of lemma 5.1. If we denote

$$\Phi_k(s_{<k}, t) := h(s,t)(1 - 2G^k_{s_{<k}}(t) + \Delta G^k_{s_{<k}}(t)) - \int_{[-1,1]} L_k(s_{<k}, s_{k-1} + u) sg(u-t) dG^k_{s_{<k}}(u) \tag{8.1}$$

the left-hand side of (5.3), then the equation becomes

$$\Phi_k(s_{<k}, t) = C_k(s_{<k}) \tag{8.2}$$

Since this equation holds for all t in a dense subset of $[-1, 1]$, then for t in $(-1, 1)$,the right and left-hand limits of $\Phi_k(s_{<k}, t)$ must be equal and this implies

$$\Delta G^k_{s_{<k}}(t)(L_k(s_{<k}, s_{k-1} + t) - h(s_{k-1}, t)) = 0 \tag{8.3}$$

A direct computation shows that this equation is equivalent to continuity in t of Φ_k, and thus (8.2) holds for all t in $(-1, 1)$. We also know by (5.2) that $\Phi_k(s_{<k}, -1) \leq C_k(s_{<k}) = \Phi_k(s_{<k}, -1_+)$ i.e.

$$(L_k(s_{<k}, s_{k-1} - 1) - h(s_{k-1}, -1))\Delta G^k_{s_{<k}}(-1) \geq 0 \tag{8.4}$$

Similarly in $t = 1$, we obtain

$$(L_k(s_{<k}, s_{k-1}+1) - h(s_{k-1}, 1))\Delta G^k_{s_{<k}}(1) \leq 0 \tag{8.5}$$

Consider at first the case $k = n$ for which $L_n(s_{\leq n}) = \varphi'(s_n)$. For all $s_{n-1} \in [-(n-1), (n-1)]$ and $t \in [-1, 1]$, $\varphi'(s_{n-1}+t)$ is bounded. Since φ' is strictly increasing we have that $\varphi'(s_{n-1}+t) - h(s_{n-1}, t) < 0$ for $t \in [-1, 0)$ and > 0 for $t \in (0, 1]$. By (8.3),(8.4) and (8.5) , $G^n_{s_{<n}}$ is continuous on these two intervals. Finally, the equalizing condition (8.2) is valid for $t = -1$ and $t = 1$ and we find :

$$h(s_{n-1}, -1) - \int_{[-1,1]} \varphi'(s_{n-1}+u)dG^n_{s_{<n}}(u) = -h(s_{n-1}, 1) + \int_{[-1,1]} \varphi'(s_{n-1}+u)dG^n_{s_{<n}}(u)$$

Since $L_{n-1}(s_{<n}) = \int_{[-1,1]} \varphi'(s_{n-1}+u)dG^n_{s_{<n}}(u)$, we conclude that:

$$C_n(s_{<n}) = \frac{1}{2}(h(s_{n-1}, -1) - h(s_{n-1}, 1)) \triangleq C(s_{n-1})$$

$$L_{n-1}(s_{<n}) = \int_{-1}^{1} \varphi'(s_{n-1}+u)\frac{du}{2} = \frac{1}{2}(\varphi(s_{n-1}+1) - \varphi(s_{n-1}-1)) \triangleq g(s_{n-1})$$

Consider next $k = n-1$, then since φ is strictly convex

$$\begin{aligned} g(s_{n-2}-1) - h(s_{n-2}, -1) &= \frac{1}{2}(\varphi(s_{n-2}) - \varphi(s_{n-2}-2)) - (\varphi(s_{n-2}) - \varphi(s_{n-2}-1)) \\ &= \varphi(s_{n-2}-1) - \frac{1}{2}(\varphi(s_{n-2}-2) + \varphi(s_{n-2})) < 0 \end{aligned}$$

By continuity we have :

$$g(s_{n-2}+t) - h(s_{n-2}, t) < 0 \text{ for } t \in [-1, \delta(s_{n-2}))$$

with $\delta(s_{n-2}) = \inf\{x \in [-1, 1] \mid g(s_{n-2}+x) - h(s_{n-2}, x) = 0\}$.

By a symmetric argument

$$g(s_{n-2}+t) - h(s_{n-2}, t) > 0 \text{ for } t \in (\gamma(s_{n-2}), 1]$$

with $\gamma(s_{n-2}) = \sup\{x \in [-1, 1] \mid g(s_{n-2}+x) - h(s_{n-2}, x) = 0\}$.

Therefore, replacing L_{n-1} by g in (8.3),(8.4), and (8.5), we deduce that $G^{n-1}_{s_{<n-1}}$ is continuous on $[-1, \delta(s_{n-2}))$ and on $[\gamma(s_{n-2}), 1]$.

The equalizing condition (8.2) is then valid for $t = -1$ and $t = 1$. This implies

$$h(s_{n-2}, -1) - \int_{[-1,1]} g(s_{n-2}+u)dG^{n-1}_{s_{<n-1}}(u) = -h(s_{n-2}, 1) + \int_{[-1,1]} g(s_{n-2}+u)dG^{n-1}_{s_{<n-1}}(u)$$

Since $L_{n-2}(s_{<n-2}) = \int_{[-1,1]} g(s_{n-2}+u)dG^{n-1}_{s_{<n-1}}(u)$, we conclude that:

$$L_{n-2}(s_{<n-2}) = \frac{1}{2}(h(s_{n-2}, 1) + h(s_{n-2}, -1)) = g(s_{n-2})$$

and

$$C_{n-1}(s_{<n-2}) = \frac{1}{2}(h(s_{n-2}, -1) - h(s_{n-2}, 1)) = C(s_{n-2})$$

All the preceding results for $k = n-1$ apply for $k = n-2$ and by induction for any $k \leq n-1$.

$\square$

PROOF OF LEMMA 5.2. Using (5.3) together integration by parts formula leads to

$$(8.6)\quad (2G^n_{s<n}(t)-1)(\varphi'(s_{n-1}+t)-h(s_{n-1},t)) = C(s_{n-1})+g(s_{n-1})-\varphi'(s_{n-1}-1) + 2\int_{[-1,t]}(2G^n_{s<n}(u)-1)d\varphi'(s_{n-1}+u)$$

Taking formally the derivative with respect to t in (5.3) gives us:

$$\frac{\partial h}{\partial t}(s_{n-1},t)(1-2G^n_{s<n}(t))+2G^n_{s<n}{}'(t)(\varphi'(s_{n-1}+t)-h(s_{n-1},t))$$
$$=(\varphi'(s_{n-1}+t)-h(s_{n-1},t))(\frac{1}{t}(1-2G^n_{s<n}(t))+2G^n_{s<n}{}'(t))=0$$

Using the boundary condition $G^n_{s<n}(-1)=0$ and $G^n_{s<n}(1)=1$, the only solution of this differential equation is $G^n_{s<n}(t)=\frac{1}{2}(1+t)$ and a direct computation shows that it is also a solution of (5.3).

To prove uniqueness, let $D(t)$ be the difference of two solutions. For $t\in[-1,0)$, it follows from (8.6) that :

$$D(t)(\varphi'(s_{n-1}+t)-h(s_{n-1},t))=\int_{[-1,t]}D(u)d\varphi'(s_{n-1}+u)$$

with $\lambda(t)=\underset{x\in[-1,t]}{sup}\frac{1}{|\varphi'(s_{n-1}+t)-h(s_{n-1},t)|}$ and $Z(t)=\underset{x\in[-1,t]}{sup}\mid D(x)\mid$, we deduce

$$\mid D(t)\mid\leq\lambda(t)Z(t)(\varphi'(s_{n-1}+t)-\varphi'(s_{n-1}-1))$$

Reporting in the first equation and with integration by parts formula:

$$\begin{aligned}\mid D(t)\mid &\leq \lambda(t)^2Z(t)\int_{[-1,t]}(\varphi'(s_{n-1}+u)-\varphi'(s_{n-1}-1))d\varphi'(s_{n-1}+u)\\ &\leq Z(t)\frac{\lambda(t)^2(\varphi'(s_{n-1}+t)-\varphi'(s_{n-1}-1))^2}{2}\end{aligned}$$

Applying this process inductively:

$$\mid D(t)\mid\leq Z(t)\frac{\lambda(t)^n(\varphi'(s_{n-1}+t)-\varphi'(s_{n-1}-1))^n}{n!}$$

which proves that $D(t)=0$ on $[-1,0)$ and by symmetry on $(0,1]$. the result follows since the solution must be nondecreasing.

□

PROOF OF LEMMA 5.3. Applying integration by parts formula in (8.2) gives us for $t\in[-1,\delta(s_{k-1}))\cup(\gamma(s_{k-1}),1]$

(8.7)
$$\begin{aligned}(2G^k_{s_{<k}}(t)-1)(g(s_{k-1}+t)-h(s_{k-1},t)) = C(s_{k-1}) + \int_{[-1,1]} g(s_{k-1}+u)dG^k_{s_{<k}}(u) - g(s_{k-1}-1)\\ + 2\int_{[-1,t]}(2G^k_{s_{<k}}(u)-1)g'(s_{k-1}+u)du\end{aligned}$$

(8.7) implies that $G^k_{s_{<k}}$ admits a derivative for
$$t \in [-1, \delta(s_{k-1})) \cup (\gamma(s_{k-1}, 1] \setminus \{0\}.$$
We recover then by differentiation equation (5.4) for these values of t :
$$\frac{\partial h}{\partial t}(s_{k-1},t)(1-2G^k_{s_{<k}}(t)) + 2{G^k_{s_{<k}}}'(t)(g(s_{k-1}+t)-h(s_{k-1},t)) = 0$$
$h(s_{k-1},\bullet)$ admits a derivative except maybe in 0 and is continuous, and thus is an absolutely continuous function. Therefore, the function
$$\frac{\frac{\partial h}{\partial x}(s_{k-1},x)}{g(s_{k-1}+x)-h(s_{k-1},x)} = \frac{\frac{1}{x}(\varphi'(s_{k-1}+x)-h(s_{k-1},x))}{g(s_{k-1}+x)-h(s_{k-1},x)}$$
is locally integrable on $[-1,\delta(s_{k-1}))$. Using the condition $G^k_{s_{<k}}(-1) = 0$, a solution of this differential equation on $[-1,\delta(s_{k-1}))$ is:

(8.8)
$$G^k_{s_{<k}}(t) = \frac{1}{2}(1 - exp(\int_{-1}^{t} \frac{\frac{1}{x}(\varphi'(s_{k-1}+x)-h(s_{k-1},x))}{g(s_{k-1}+x)-h(s_{k-1},x)}dx))$$

If $\delta(s_{k-1}) \neq 0$, $\frac{1}{x}(\varphi'(s_{k-1}+x)-h(s_{k-1},x))$ being positive and since $g(s_{k-1}+x)-h(s_{k-1},x)$ admits a derivative in $\delta(s_{k-1})$:
$$\int_{-1}^{t} \frac{\frac{1}{x}(\varphi'(s_{k-1}+x)-h(s_{k-1},x))}{g(s_{k-1}+x)-h(s_{k-1},x)}dx \underset{t\to\delta(s_{k-1})^-}{\longrightarrow} -\infty \Longrightarrow G^k_{s_{<k}}(t) \underset{t\to\delta(s_{k-1})^-}{\longrightarrow} \frac{1}{2}$$
If φ is C^2 with $\varphi'' > 0$, then this result is true even if $\delta(s_{k-1}) = 0$ since in this case $\frac{\partial h}{\partial x}(s_{k-1},x)$ is continuous and positive on $[-1,1]$.

A symmetric argument shows that:

(8.9)
$$G^k_{s_{<k}}(t) = \frac{1}{2}(1 + exp(-\int_{t}^{1} \frac{\frac{1}{x}(\varphi'(s_{k-1}+x)-h(s_{k-1},x))}{g(s_{k-1}+x)-h(s_{k-1},x)}dx)) \text{ for } t \in (\gamma(s_{k-1}),1]$$

If $\gamma(s_{k-1}) \neq 0$ or φ is C^2 with $\varphi'' > 0$, $G^k_{s_{<k}}(t) \underset{t\to\gamma(s_{k-1})^+}{\longrightarrow} \frac{1}{2}$.

We can check from (8.2) that a solution is then

(8.10)
$$G^k_{s_{<k}}(t) = \begin{cases} \frac{1}{2}(1 - exp(\int_{-1}^{t} \frac{\frac{1}{x}(\varphi'(s_{k-1}+x)-h(s_{k-1},x))}{g(s_{k-1}+x)-h(s_{k-1},x)}dx)) & t \in [-1,\delta(s_{k-1})) \\ \frac{1}{2} & t \in [\delta(s_{k-1}),\gamma(s_{k-1})) \\ \frac{1}{2}(1 + exp(-\int_{t}^{1} \frac{\frac{1}{x}(\varphi'(s_{k-1}+x)-h(s_{k-1},x))}{g(s_{k-1}+x)-h(s_{k-1},x)}dx)) & t \in [\gamma(s_{k-1}),1] \end{cases}$$

and that $G^k_{s_{<k}}$ is solution of the differential equation (5.4) a.e. $t \in [-1,1]$. Since this solution depends only on s_{k-1}, it will be denoted $G_{s_{k-1}}$. This particular form of the conditional distributions implies that $(S_k)_{k=1,..,n-1}$ is a time-homogeneous Markov chain.

To prove uniqueness, let $D(t)$ be the difference of two solutions. For $t \in [-1, \delta(s_{k-1}))$, it follows from (8.7) that :

$$D(t)(g(s_{k-1}+t) - h(s_{k-1}, t)) = \int_{[-1,t]} D(u)dg(s_{k-1}+u)$$

with $\lambda(t) = \underset{x\in[-1,t]}{sup} \frac{1}{|g(s_{k-1}+t)-h(s_{k-1},t)|}$ and $Z(t) = \underset{x\in[-1,t]}{sup} \mid D(x) \mid$, we deduce

$$\mid D(t) \mid \leq \lambda(t)Z(t)(g(s_{k-1}+t) - g(s_{k-1}-1))$$

Reporting in the first equation and with integration by parts formula:

$$\begin{aligned} \mid D(t) \mid &\leq \lambda(t)^2 Z(t) \int_{[-1,t]} (g(s_{k-1}+u) - g(s_{k-1}-1))dg(s_{k-1}+u) \\ &\leq Z(t)\frac{\lambda(t)^2(g(s_{k-1}+t) - g(s_{k-1}-1))^2}{2} \end{aligned}$$

Applying this process inductively:

$$\mid D(t) \mid \leq Z(t)\frac{\lambda(t)^n(g(s_{k-1}+t) - g(s_{k-1}-1))^n}{n!}$$

which proves that $D(t) = 0$ on $[-1, \delta(s_{k-1}))$ and by symmetry on $(\gamma(s_{k-1}), 1]$. Right-continuity of the solutions implies also the equality in $t = \gamma(s_{k-1})$.

The only cases where (8.8) and (8.9) do not define uniquely a distribution function is when

$$\delta(s_{k-1}) = 0,\ \gamma(s_{k-1}) > 0,\ G^k_{s_{<k}}(t) \underset{t\to 0-}{\longrightarrow} d < \frac{1}{2}$$

and the symmetric case. Suppose in this case that we have two different solutions A, B, where A is given by (8.10), these solutions must coincide on $[-1, 0)$ and $[\gamma(s_{k-1}), 1]$ and we have:

$$h(s_{k-1}, t)(1 - 2A(t) + \Delta A(t)) - \int_{[-1,1]} g(s_{k-1}+u)sg(u-t)dA(u) = C(s_{k-1})$$
$$h(s_{k-1}, t)(1 - 2B(t) + \Delta B(t)) - \int_{[-1,1]} g(s_{k-1}+u)sg(u-t)dB(u) = C(s_{k-1})$$

Subtracting these two equation for some $t \in [-1, 0)$, we find

$$\int_{[0,\gamma(s_{k-1})]} g(s_{k-1}+u)dB(u) = \int_{[0,\gamma(s_{k-1})]} g(s_{k-1}+u)dA(u) = g(s_{k-1})\Delta A(0)$$

But since $g(s_{k-1} + \bullet)$ is strictly increasing and measures dA and dB give the same mass to $[0, \gamma(s_{k-1})]$, this implies $B = A$ on $[0, \gamma(s_{k-1})]$.

□

We end this section with an useful lemma

LEMMA 8.1: *Let S and S' be polish spaces and h_n and h Borel-measurable functions from S to S'. Let P_n be a sequence of probabilities over S weakly converging to P. Define E as the subset of $x \in S$ such that there exists a sequence $x_n \to x$ such that $h_n(x_n)$ does not converge to $h(x)$ and let $P \circ h^{-1}$ denote the image probability of P induced by the mapping h.*
Using the former notations, if $P(E) = 0$, then $P_n \circ h_n^{-1}$ converges to $P \circ h^{-1}$.

PROOF. See [**10**] theorem 5.5 p.33 □

Extension to integrable distributions. In order to extend the results of existence of equilibria in the case of integrable distributions over $\mathbb{R}$ (denoted $\Delta^1(\mathbb{R})$ hereafter), we have to modify the definition of admissible strategies and the set of possible dual functions ψ accordingly. The set of admissible strategies of P1 in this case is the set of strategies σ such that the distribution π induced by the pair (μ, σ) on $(L, p_1, .., p_n)$ is in $\Delta^1(\mathbb{R}^{n+1})$. The set of admissible strategies of P1 in the dual game is then $\Delta^1(\mathbb{R}^{n+1})$. For P2, an admissible strategy τ is such that for any k and any $p_{<k} \in \mathbb{R}^{k-1}$, $\tau_k(p_{<k}) \in \Delta^1(\mathbb{R})$. With these definitions, for any pair of admissible strategies, the payoff function of P1 is well defined in $\mathbb{R}$ and the payoff function of P2 well-defined in $\mathbb{R} \cup \{-\infty\}$.

The definition of the reduced strategy $\tau(\varphi)$ in section 4 is still meaningful for any convex function φ with real values on $(-n, n)$ and defines an admissible strategy. The only problem is to specify the class of function D_n for which the associated strategy of P1 is admissible. The definition of the process S in section 5 requires only that the function φ is C^1 on $(-n, n)$ and such that φ' is strictly increasing on this interval. The associated strategy of P1 is admissible if and only if $L = \varphi'(S_n) \in L^1$ and these conditions define the set D_n. It seems difficult however to obtain a simple expression for this set except for $n = 1$ where this condition becomes $\varphi'(U) \in L^1$, where U is uniformly distributed on $[-1, 1]$.

However, this extension is still expressed in terms of the dual game and the main difficulty to prove the conjecture we addressed in the introduction is to invert this link. We know that for a fixed probability μ with compact support as in the conjecture, we can find a sequence of functions (ψ_n), such that the sequence of equilibria in $G^*_n(\psi_n)$ induce a sequence of equilibria in $G_n(\mu)$. But the question of existence only is not sufficient, in order to use the result we have proved here, we still need to show that the sequence ψ_n is not far, in a sense that has to be made precise, from the sequence used in the proposition 6.1.

ANNEXE A

1. Probabilités, métriques, lois conditionelles.

Soit E un espace topologique et $\mathcal{B}(E)$ la tribu borélienne sur E. On note $\Delta(E)$ l'espace des probabilités sur $\mathcal{B}(E)$. Dans le corps du manuscrit, les espaces E considérés sont toujours métriques séparables et la plupart du temps polonais, c'est à dire dont la topologie est métrisable par une métrique d qui rend (E, d) complet et séparable.

On munira implicitement tout espace topologique de la tribu borélienne associée, et tout produit d'espaces topologiques de la topologie produit.

NOTATION 13: *Si E, F sont deux espaces métriques séparables, et Q,R deux sous-ensembles de $\Delta(E)$ et $\Delta(F)$. Alors on note $\mathcal{P}(Q, R)$ l'ensemble des probabilités sur $E \times F$ dont les distributions marginales sur X et Y appartiennent respectivement à Q et R. Si $Q = \{\mu\}$, on écrira simplement $\mathcal{P}(\mu, R)$.*

LEMME 1.1: *Si E, F sont deux espaces métriques séparables et si Q,R sont des sous-ensembles tendus (resp. fermés, convexes) de $\Delta(E)$ et $\Delta(F)$. Alors l'ensemble $\mathcal{P}(Q, R)$ est lui même tendu (resp. fermé, convexe).*

DÉFINITION 1.1: *On définit la topologie faible w sur $\Delta(E)$ comme la topologie la moins fine rendant continue toutes les applications de la forme*

$$\mu \in \Delta(E) \longmapsto \int f d\mu \in \mathbb{R}$$

pour toute fonction $f \in C_b(E)$ (continue bornée).

LEMME 1.2: $(\Delta(X), w)$ *est un espace polonais.*

DÉFINITION 1.2: *Dans la cas particulier où $X = \mathbb{R}^d$, pour $p \in [1, \infty)$, on definit sur le sous-ensemble*

$$\Delta^p(\mathbb{R}^d) = \{\mu \in \Delta(\mathbb{R}^d) \,:\, \int_{\mathbb{R}^d} |x|^p d\mu < \infty\}$$

La topologie W_p associée à la distance de Wasserstein d'ordre p par

$$d_{W_p}(\mu, \nu) = \min_{\pi \in \mathcal{P}(\mu,\nu)} (\int |y - x|_p^p d\pi(x, y))^{\frac{1}{p}} = min\{\|X - Y\|_{L^p} \mid X \sim \mu \ , \ Y \sim \nu\}$$

LEMME 1.3: *(voir proposition 7.1.5 dans* [**2**] *et remarque 7.1.11)*
L'espace métrique $(\Delta^p(\mathbb{R}^d), d_{W_p})$ est complet et séparable (donc polonais) . La topologie W_p coincide avec la topologie engendrée par l'ensemble $C_p(\mathbb{R}^d)$ des fonctions continues à croissance au plus polynomiale d'ordre p, i.e. la topologie la moins fine rendant continues les applications

$$\mu \in \Delta(X) \longmapsto \int f d\mu \in \mathbb{R}$$

pour $f \in C_p(\mathbb{R}^d)$. En particulier, une suite μ_n W_p-converge vers μ si et seulement si μ_n converge faiblement vers μ et les moments d'ordre p convergent, c'est à dire

$$\int_{\mathbb{R}^d} |x|^p d\mu_n \longrightarrow \int_{\mathbb{R}^d} |x|^p d\mu$$

Un sous-ensemble S de $\Delta^p(\mathbb{R}^d)$ est relativement compact si et seulement si il admet des moments d'ordre p uniformément intégrables, i.e.

$$\lim_{M\to\infty} \sup_{\mu\in S} \int_{\mathbb{R}^d} |x|^p \mathbb{1}_{|x|\geq M} d\mu(x) = 0$$

Ce dernier critère s'applique en particulier si S a des moments d'ordre $q > p$ uniformément bornés. Sur un tel ensemble, les topologies w et W_p coincident.

On utilisera en particulier le lemme suivant (lemme 5.2.4 dans [**2**]).

Lemme 1.4: *Soient $X = Y = \mathbb{R}^d$ et $\pi_n \in \mathcal{P}(X \times Y)$ est une suite faiblement convergente de limite π telle que*

$$\sup_n \int |x|^p + |y|^q d\pi_n(x,y) < \infty \quad \text{pour } p,q \in (1,\infty) \text{ tels que } \frac{1}{p} + \frac{1}{q} = 1.$$

Si la suite de lois marginales μ_n sur X a des moments d'ordre p uniformément intégrables (resp. ν_n sur Y a des moments d'ordre q uniformément intégrables) alors

$$\int \langle x,y\rangle d\pi_n(x,y) \underset{n\to\infty}{\longrightarrow} \int \langle x,y\rangle d\pi(x,y)$$

Définition 1.3: *Etant donné Y polonais, on définit sur*

$$\Delta^1(\mathbb{R}^d \times Y) = \{\mu \,:\, \int_{\mathbb{R}^d\times Y} |x| d\mu(x,y) < \infty\}$$

une topologie mixte W_1, w comme la topologie la moins fine rendant continue les applications

$$\mu \longrightarrow \int_{\mathbb{R}^d\times Y} g(x,y) d\mu(x,y)$$

pour $g : \mathbb{R}^d \times Y \to \mathbb{R}$ continue et telle qu'il existe une constante C vérifiant

$$\forall y \in Y, \quad g(x,y) \leq C(1 + |x|)$$

remarque 1.1: *Toutes les topologies considérées engendrent les mêmes tribus boréliennes que la tribu induite par la tribu borélienne associée à la topologie faible w. On peut munir par exemple $\Delta^p(\mathbb{R}^d)$ des topologies w et W_q pour $1 \leq q \leq p$. Ces topologies étant de plus en plus fines, on a une inclusion directe. Pour l'autre inclusion, on utilise la semi-continuité inférieure de d_{W_p} sur $(\Delta^p(\mathbb{R}^d), w)$ (proposition 7.1.3 dans [**2**]). Ce résultat vaut aussi pour la topologie mixte avec la même preuve car cette topologie est en fait une distance de Wasserstein d'ordre 1 associée à une métrique produit $|| + d$ sur $\mathbb{R}^d \times Y$ où d est une distance bornée compatible avec la topologie de Y.*

On rappelle la définition classique des transitions

Définition 1.4: *Soient $(E, \mathcal{E})$ un espace mesurable et F un espace polonais. Une probabilité de transition de $(E, \mathcal{E})$ dans F est une famille de probabilités sur F indexée par $x \in E$, $q((. \mid x))_{x\in E}$ telle que pour tout $B \in \mathcal{B}(Y)$, l'application $x \to q(B \mid x)$ est $\mathcal{E}$-mesurable.*

et la caractérisation que nous utilisons tout au long du manuscrit.

LEMME 1.5: *Si E et F sont polonais, alors $q((.|x))_{x\in E}$ est une probabilité de transition de E dans F si et seulement si l'application $x \to q(.|x)$ est mesurable de E dans $\Delta(F)$.(proposition 7.26 dans* [**9**]*).*

Passons maintenant à la définition des lois conditionnelles

DÉFINITION 1.5: *Soient X une variable aléatoire à valeurs dans un espace topologique E définie sur un espace de probabilité $(\Omega, \mathcal{A}, \mathbb{P})$ et $\mathcal{F}$ une sous-tribu de $\mathcal{A}$. On appelle loi conditionnelle (ou version régulière de la loi conditionnelle) de X sachant $\mathcal{F}$ une probabilité de transition $q(.|\omega)$ de $(\Omega, \mathcal{F})$ dans E telle que pour toute fonction f mesurable bornée*

$$\int_{\Omega} f(X(\omega))d\mathbb{P}(\omega) = \int_{\omega} \left(\int_{E} f(x)q(dx \mid \omega) \right) d\mathbb{P}(\omega)$$

THÉOREME 1.1: *(Désintégration, voir 1.2 p.65 dans* [**36**]*) Dans la définition précédente, si E est polonais, alors il existe une loi conditionnelle de X sachant $\mathcal{F}$.*

Citons maintenant une version paramétrée du théorème de désintégration.

THÉOREME 1.2: *(proposition 7.27 dans* [**9**]*)*
Soient E, F, G trois espaces polonais, et $r(.|x)$ une probabilité de transition de E dans $F \times G$. Alors il existe des probabilités de transition $q(.|x, y)$ de $E \times F$ dans G et $p(.|x)$ de E dans F telles que

$$\forall (A, B) \in \mathcal{B}(F), \mathcal{B}(G), \quad r(A \times B|x, y) = \int_{A} q(B|x, y)p(dy|x)$$

On rappelle enfin le théorème classique suivant, qui permet de construire des variables aléatoires avec des lois conditionnelles prescrites.

THÉOREME 1.3: *(Blackwell-Dubins* [**13**]*)*
Soit E un espace polonais et $([0, 1], \mathcal{B}([0, 1]), \lambda)$ l'intervalle unité muni de la mesure de Lebesgue. Il existe une application mesurable

$$\Phi : [0, 1] \times \Delta(E) \longrightarrow E$$

telle que pour toute loi $\mu \in \Delta(E)$, la loi de $\Phi(U, \mu)$ est exactement μ en notant U l'élément canonique $[0, 1]$ (où encore si U est une variable de loi uniforme sur $[0, 1]$). On fait parfois appel à ce résultat sans mentionner l'espace E concerné et en gardant la notation Φ pour différents espaces.

On utilise ce résultat combiné avec un résultat de désintégration dans les lemmes 4.1 et 6.1 du chapitre 2. Précisément

LEMME 1.6: *Soient X, Y des variables aléatoires à valeurs dans E définies sur un même espace de probabilités, U une variable uniforme sur $[0, 1]$, indépendante de (X, Y) et f une application mesurable de E dans $\Delta(E^2)$. Soit $f_1(x)$ la loi marginale de $f(x)$ induite sur la première coordonnée. Si $f_1(X)$ est une version de la loi conditionnelle de Y sachant X, alors il existe une variable aléatoire $Z = \varphi(X, Y, U)$ telle que $f(X)$ est une version de la loi conditionnelle de (Y, Z) sachant X.*

Démonstration. On peut désintégrer par le théorème 1 la loi jointe $f(x)$ en une paire $f_1(x), g(x,y)$ où f_1 est la loi marginale donnée dans l'énoncé et g une application à valeur dans $\Delta(E)$. La variable $Z = \Phi(g(X,Y),U)$ possède alors les propriétés requises. $\square$

Le résultat précédent veut simplement dire que l'on peut construire une variable aléatoire vectorielle (Y,Z) ayant une loi conditionelle sachant X fixée sur un espace de probabilités où X et Y sont déjà construites si la loi du couple X,Y est compatible avec la loi conditionnelle choisie.

2. Théorèmes de selection.

On rappelle ici les trois énoncés principaux de théorèmes de selection mesurables utilisés au cours de la thèse.

Théoreme 2.1: *(proposition 7.33 dans* [**9**]*)*
Soient E métrique F compact métrique, D fermé dans $E \times F$ et f s.c.s. sur D à valeurs dans $\overline{\mathbb{R}}$. On pose pour x dans D_E (projection de D sur E)

$$g(x) = \underset{y\,:\,(x,y)\in D}{sup} f(x,y)$$

Alors D_E est fermé, g est s.c.s. sur D_E et il existe une fonction borélienne ϕ de D_E dans F dont le graphe est inclus dans D et telle que

$$\forall x \in D_E, \quad f(x,\phi(x)) = g(x)$$

Théoreme 2.2: *(proposition 7.34 dans* [**9**]*)*
Soient E métrique, F métrique séparable, D ouvert dans $E \times F$ et f s.c.i. sur D on pose

$$g(x) = \underset{y\,:\,(x,y)\in D}{sup} f(x,y)$$

Alors D_E est ouvert, g est s.c.i. sur D_E et pour tout $\varepsilon > 0$, il existe une fonction borélienne ϕ_ε de D_E dans F dont le graphe est inclus dans D et telle que

$$\forall x \in D_E, \quad f(x,\phi(x)) \begin{cases} \geq g(x) - \varepsilon & si \quad g(x) < +\infty \\ \geq \frac{1}{\varepsilon} & si \quad g(x) = +\infty \end{cases}$$

Définition 2.1: *Soient E et F polonais. On rapelle que la tribu universellement mesurable sur E, $\mathcal{U}(E)$ est définie par*

$$\mathcal{U}(E) = \bigcap_{\mu\in\Delta(E)} \mathcal{B}(E)^\mu$$

où $\mathcal{B}(E)^\mu$ est la complétion de $\mathcal{B}(E)$ par rapport à μ (la tribu engendrée par $\mathcal{B}(E)$ et les sous-ensembles μ-négligeables de E). Une application de E dans F est universellement mesurable si

$$\forall B \in \mathcal{B}(F),\ f^{-1}(B) \in \mathcal{U}(E)$$

Lemme 2.1: *(lemme 7.27 dans* [**9**]*)*
Soient E et F polonais. Soit $\mu \in \Delta(E)$ et f universellement mesurable de E dans F, alors il existe g borélienne de E dans F telle que $\{f \neq g\}$ soit μ-négligeable.

Théoreme 2.3: *(Von Neumann, proposition 7.49 dans* [**9**]*)*
Soient E et F polonais. Si D est Borel dans $E \times F$ alors il existe une application universellement mesurable de D_E dans F dont le graphe est inclus dans D.

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